7. Numbers

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7.1 Properties of Numbers

Properties of Numbers

- For us, *real* numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers, though real numbers may be understood as a proper subset of the complex numbers.
- There are many subsets of real numbers that are familiar to us; we want to understand their properties.
- Most quantities used to describe things in the world may be understood as real numbers.

Integers

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

$$..., -2, -1, 0, 1, 2, ...$$

Rational Numbers

- Rational numbers are *ratios/fractions* of integers.
- Any number of the form $\frac{p}{q}$ for p, q, integers is rational.
- In particular, every integer is also considered a rational number.
- One must take care: $q = 0$ is not permitted, as this involves division by 0.
- Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.

Irrational Numbers

- **There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q.**
- **Such numbers are called *irrational*; there are many of them.**
- **Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$.**
- **These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat.***

Identify each of the following numbers as integer, rational, or irrational

$2 + \sqrt{2}$ | $\frac{12}{6}$ | $\frac{121}{11}$

π | $\frac{\pi}{3\pi}$ | $-\frac{16}{3}$

$\frac{17}{4}$ | $\sqrt{49} + \sqrt{64}$ | e^0

Label as true or false:

- Every rational number is an integer.
- There are integers that are irrational.
- Every integer is rational.

Algebraic numbers are those that are roots of polynomials with integer coefficients. Give an example of a number that is irrational but algebraic.

7.2 Elementary Number Theory

Elementary Number Theory

- Number theory concerns properties of integers.
- It is one of the oldest mathematical subjects, and many of its most famous unsolved questions are rather easy to state.
- Indeed, number theory is well-known amongst mathematicians for having very hard answers to very simple-to-state question.

Divisibility

- We say an integer p *divides* an integer q if $q = p \times r$ for some integer r .
- Numbers that divide q are called the *divisors* of q .
- For any integer p , it is a simple exercise to show that both $1 \text{ and } p$ are divisors of p. If these are the only divisors, we call p a prime number.
- Primes are quite mysterious. There ancient mathematical problems related to them that are still unsolved.

Find the greatest common divisor of the following pairs:

$(20, 42)$

$(30, 36)$

$(9, 72)$

$(8, 24)$

Even and Odd Integers

- The integers $\ldots, -4, -2, 0, 2, 4, \ldots$ are the *even* numbers.
- The remaining integers $\ldots, -3, -1, 1, 3, \ldots$ are the *odd* integers.
- One can define these in terms of divisors.
- The even numbers are the integers having 2 as a divisor.
- The odd numbers are the integers that do not have 2 as a divisor.

Show that the sum of any two odd integers is even.

Show that the product of any two odd integers is odd.

Fundamental Theorem of Arithmetic

- **Sometimes called the Unique Prime Factorization Theorem.**

- **It is a classic result that has ancient roots.**

Every integer n may be written uniquely as

$$n = p_1^{m_1} p_2^{m_2} \ldots, p_r^{m_r}$$

for prime numbers $p_1, \ldots, p_r$

and positive integers $m_1, \ldots, m_r$

Find the prime factorization of the following integers:

8	17	100
15	25	2

7.3 Scientific Notation and Unit Conversion

Scientific Notation and Unit Conversion

- This submodule addresses important ideas for scientific computation.
- *Scientific notation* is simply an efficient way of writing numbers with many zeros.
- For very large or very small numbers, it is more compact than traditional notation.

- A number is written in scientific notation by writing it as

$$x = a \times 10^n$$

- Here, a is between $1 \text{ and } 10$.
- If $x > 10, \text{ then } n \geq 1.$
- If $0 < x < 1, \text{ then } n \leq -1.$
- The exponent n counts the number of zeroes before the decimal point if positive, and the number of zeroes after the decimal point if negative.
- One may do algebra in scientific notation, by following the usual product and exponent rules.

Convert to scientific notation:

$.0007$

-74561

2310

$-.003$

Unit Conversion

- Many applications of mathematics require shifting between two different measurement systems, such as feet and meters, gallons and liters, Fahrenheit and Celsius.

- The rule for conversion is simple:

$$\text{Unit A} \xrightarrow{\times \frac{\text{Unit B}}{\text{Unit A}}} \text{Unit B}$$

Convert units:

$$100\,\frac{\text{km}}{\text{hour}} \text{ to } \frac{\text{miles}}{\text{minute}}$$

$$5\,\frac{\text{US dollars}}{\text{gallon}} \text{ to } \frac{\text{Euros}}{\text{liter}}$$

$$12\,\frac{\text{feet}}{\text{second}} \text{ to } \frac{\text{inches}}{\text{minute}}$$

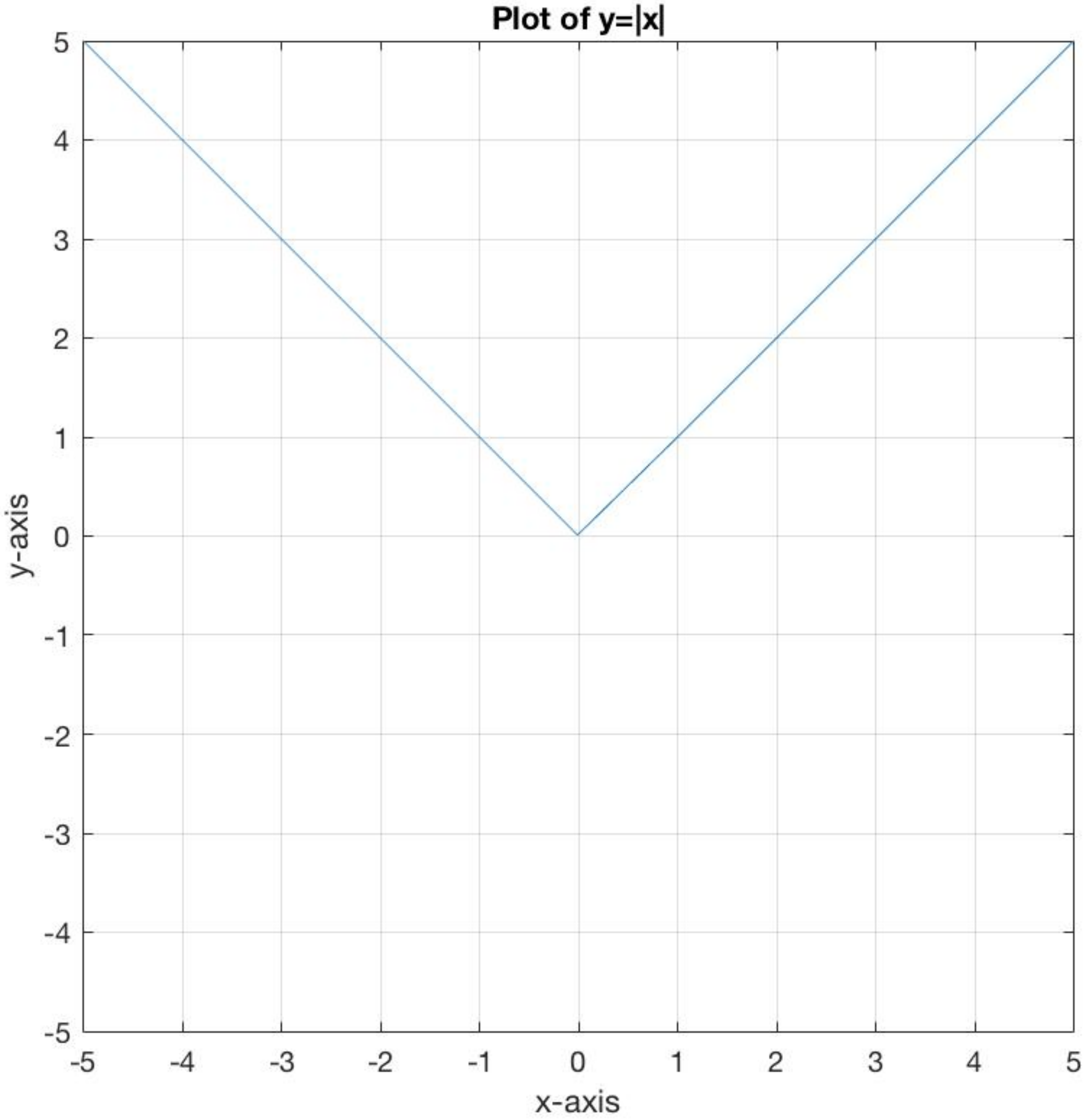
7.4 Absolute Value

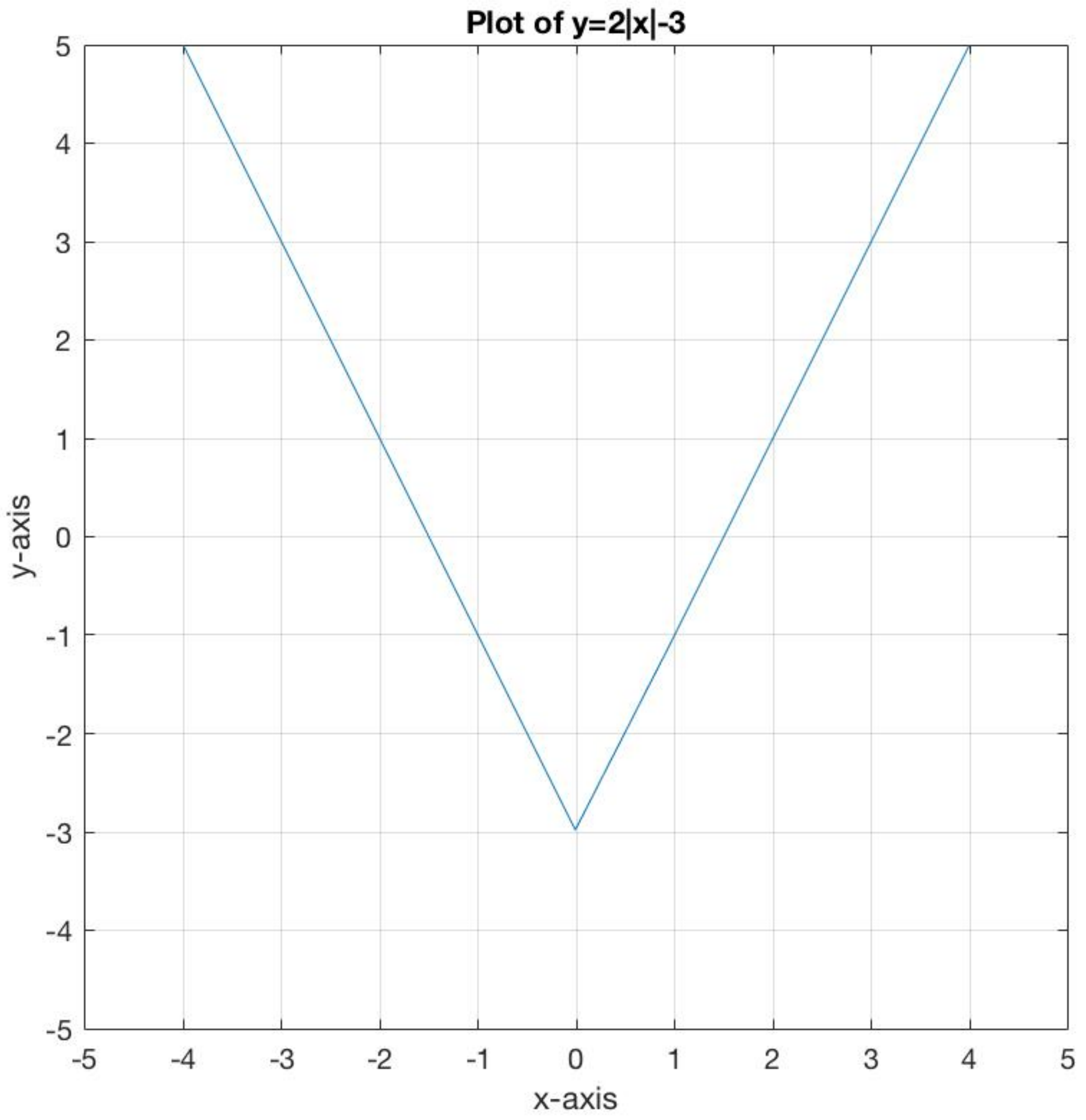
Absolute Value

Recall the absolute value function, which is equal to a number's distance from 0:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, the absolute value function keeps positive numbers the same, and switches negative numbers into their positive counterpart.


Plot of y=|x|
5
4
3
2
1
0
-1
-2
-3
-4
-5
y-axis
-5
-4
-3
-2
-1
0
1
2
3
4
5
x-axis



Plot of y=2|x|-3
y-axis
x-axis
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Evaluate $f(2)$ if f has the following formulae:

$$f(x) = |x+1|$$

$$f(x) = -|x-3|$$

$$f(x) = |x-7|$$

$$f(x) = |x-2|$$

Equations with Absolute Value

When considering equations of the form: $|f(x)| = g(x)$,

it suffices to consider the two cases

$$f(x) = g(x) \text{ and } -f(x) = g(x)$$

In the case of absolute value equations involving first order polynomials (linear functions), we get:

$$|ax+b| = c \Leftrightarrow ax+b = c \text{ or } -(ax+b) = c$$

Solve $|2x - 3| = 1$

Inequalities Involving Absolute Values

When considering systems of absolute value *inequalities*, great care must be taken.

In general,

$$|f(x)| \leq g(x) \Leftrightarrow \begin{cases} f(x) \leq g(x) \text{ and } f(x) > 0 \\ -f(x) \leq g(x) \text{ and } f(x) \leq 0 \end{cases}$$

A similar equivalence holds for $|f(x)| \geq g(x)$.

Linear Absolute Value Inequalities

- **One can, when working with inequalities of the form**

$$|ax+b| \leq c \text{ or } |ax+b| \geq c$$

proceed by finding the two solutions to

$$ax+b=c \text{ and } -(ax+b)=c$$

- **These can then be plotted on a number line, and checking in which region the desired inequality is achieved. This is the *number line method.***

Solve $|x+3| \leq 2$