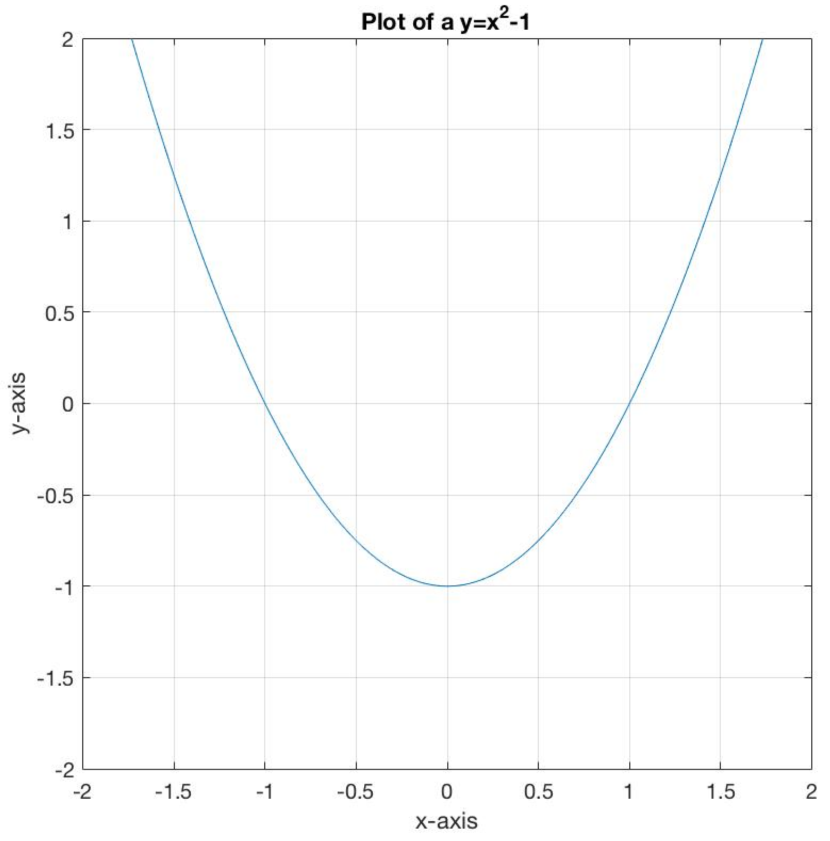
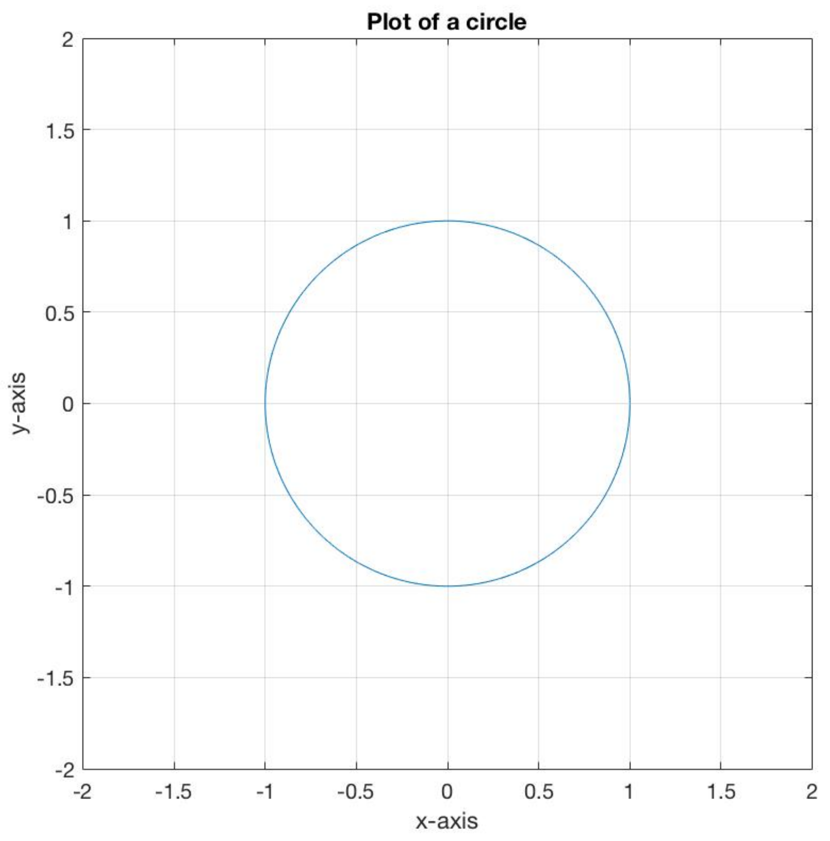
3.1.1 What is a Function?

- Functions are mathematical objects that send an input to a unique output.
- They are often, but not always, numerical.
- The classic notation is that $f(x)$ denotes the output of a function f at input value x.
- Functions are abstractions, but are very convenient for drawing mathematical relationships, and for analyzing these relationships.

3.1.2 Function or not?

One of the key properties of a function is that it assigns a unique output to an input.

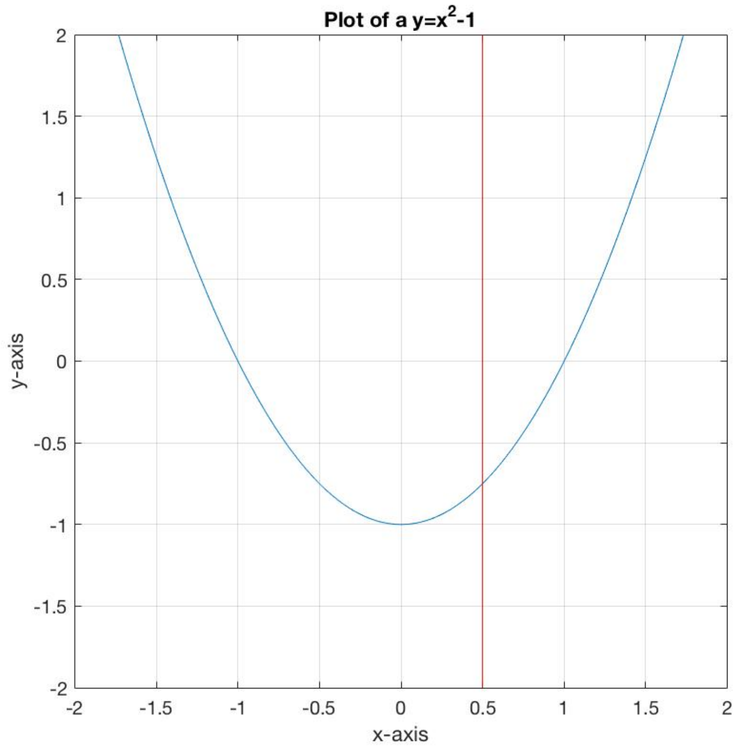


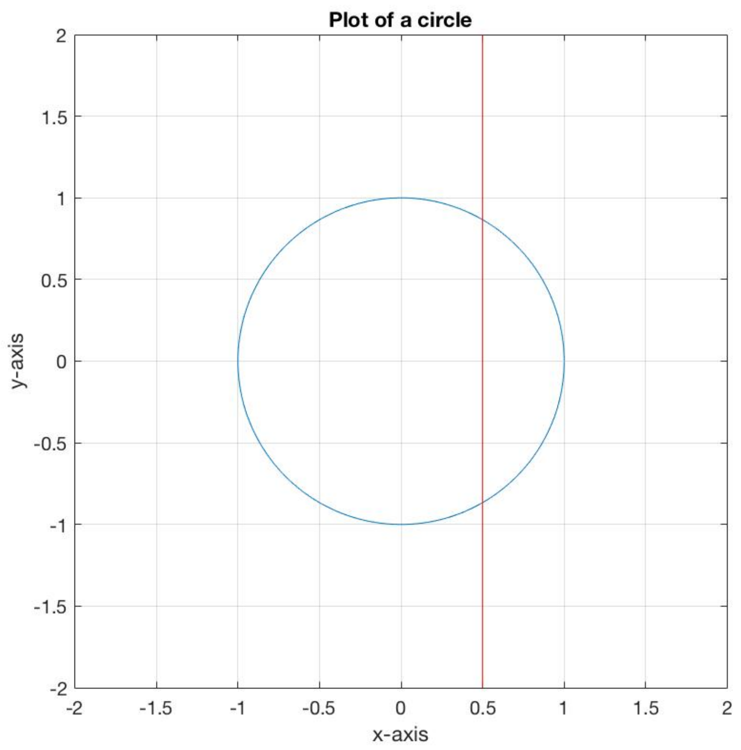


3.1.3 Vertical Line Test

A trick for checking if a mathematical relationship plotted in the Cartesian plane is a function is the *vertical line test.*

VLT: A plot is a function if and only if every vertical line intersects the plot in at most one place.





3.2.0 Representing with Functions

Functions are convenient for describing numerical relationships.

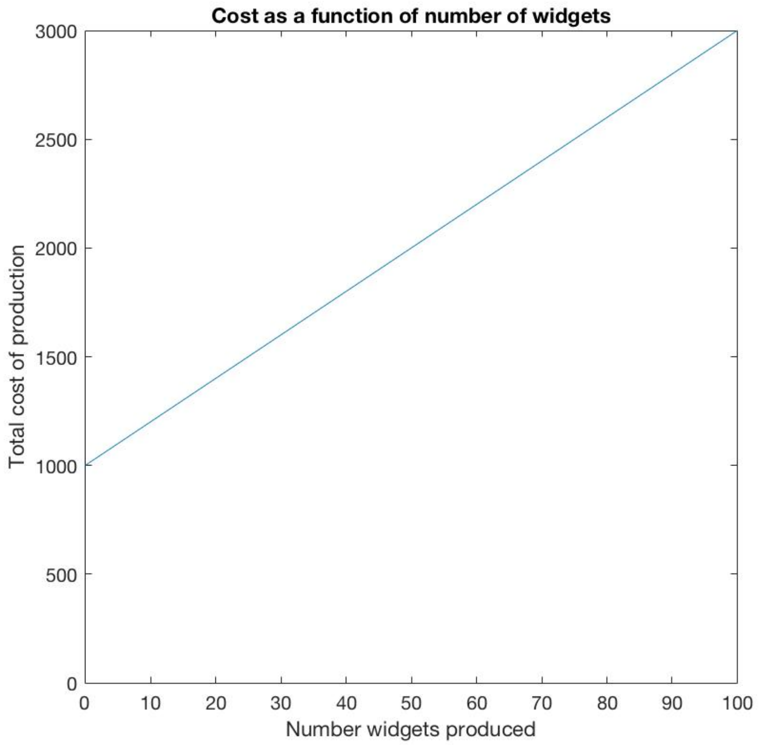
$$\text{Input} \xrightarrow{f} \text{Output}$$

To model a relationship with functions, you simply need to understand how your input depends on your output.

3.2.1 Linear Modeling

Some simple relationships can be modeled with *linear relationships* of the form $f(x) = ax + b$

For example, suppose the cost of producing a widgets is $1000, plus $20 for each widget produced. Then the total cost of producing x widgets is modeling as $C(x) = 20x + 1000$

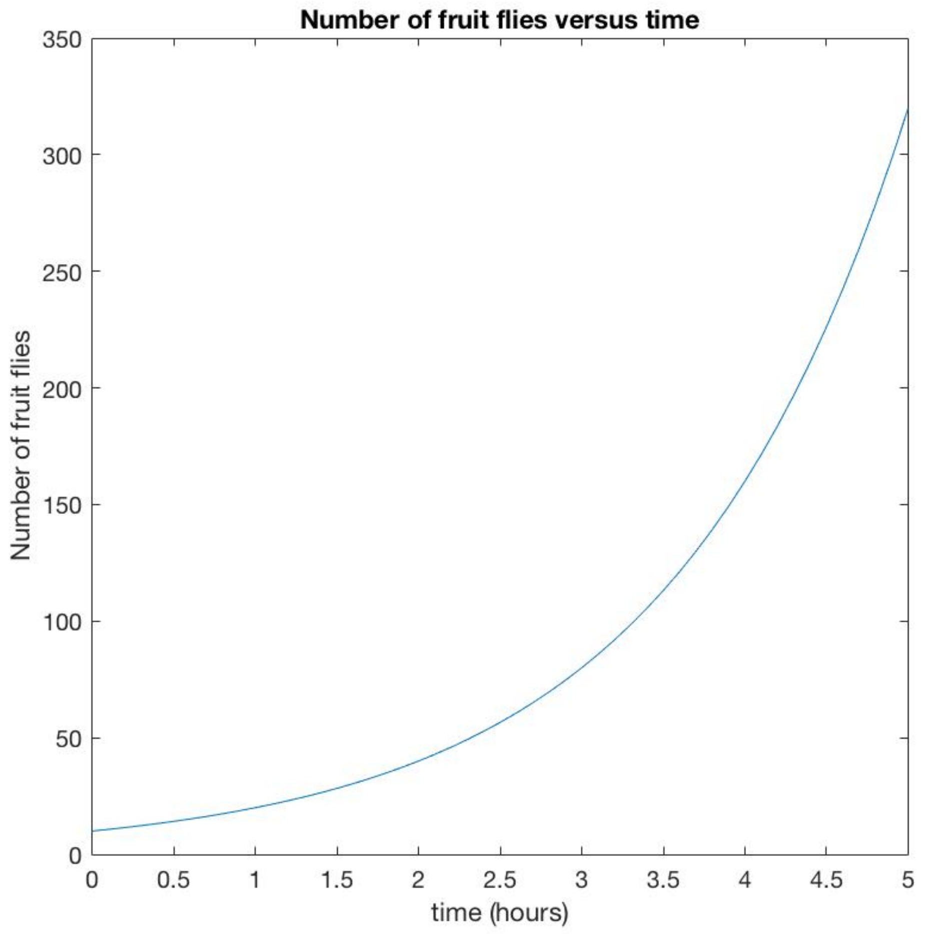


3.2.2 Exponential Modeling

Exponential functions are more complicated than linear functions, but are very useful for things that, for example double in magnitude at a certain rate.

For example, suppose a colony of fruit flies starts with 10, and doubles every hour. Then the population of fruit flies at time t in hours is given as

$$P(t) = 10 \cdot 2^t$$



3.3.0 Domain and Range of a Function

Let $f(x)$ be a function.

- **The *domain* of $f(x)$ is the set of allowable inputs.**
- **The *range* of $f(x)$ is the set of possible outputs for the function.**
- **These can depend on the relationship the functions are modeling, or be intrinsic to the mathematical function itself.**
- **They can also be inferred from the plot of $f(x)$, if it is available.**

3.3.1 Intrinsic Domain Limitations

Some mathematical objects have intrinsic limitations on their domains and ranges. Classic examples include:

- $f(x) = x^2$ **has domain** $(-\infty, \infty)$**, range** $[0, \infty)$ **.**
- $f(x) = \sqrt{x}$ **has domain** $[0, \infty)$ **, range** $[0, \infty)$**.**
- $f(x) = \log(x)$ **has domain** $(0, \infty)$ **, range** $(-\infty, \infty)$**.**
- $f(x) = a^x$ **has domain** $(-\infty, \infty)$ **, range** $(0, \infty)$ **.**
- $f(x) = \frac{1}{x}$ **has domain and range** $(\infty, 0) \cup (0, \infty)$ **.**

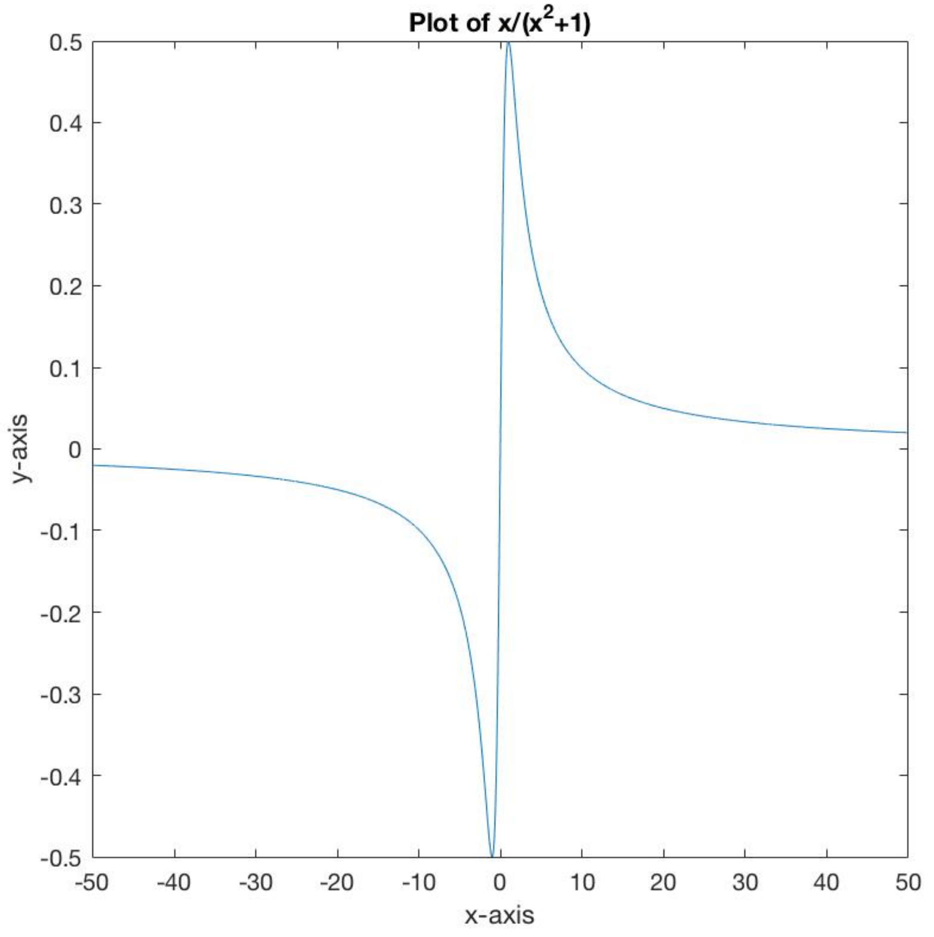
3.3.2 Visualizing Domain and Range

Given a plot of $f(x)$, one can observe the domain and range by considering what x and y values are achieved.

The function

$$f(x) = \frac{x}{x^2+1}$$

is hard to analyze, but its plot helps us guess its domain and range.



3.4.0 Algebra of Functions

- **Functions may be treated as algebraic objects: they may added, subtracted, multiplied, and divided in natural ways.**
- **One must take care in dividing by functions that can be 0. Division by 0 is not defined.**
- **There is one important operation of functions that does not apply to numbers: the operation of *composition.***
- **In essence, composing functions means applying one function, then the other.**

3.4.1 Composition of Functions

Given two functions $f(x), g(x)$, the *composition of* $f(x)$ *with* $g(x)$ is denoted $(f \circ g)(x)$, and is defined as:

$$(f \circ g)(x) = f(g(x)) \; .$$

Similarly, $(g \circ f)(x) = g(f(x))$.

One thinks of $(f \circ g)(x)$ as first applying the rule $g(x)$, then applying the rule $f(x)$.

As an example, consider $f(x) = x + 1, g(x) = x^2$. By substituting $g(x)$ into $f(x)$, one sees that

$$(f \circ g)(x) = x^2 + 1$$

Similarly, one can substitute $f(x)$ into $g(x)$ to compute that

$$(g \circ f)(x) = (x + 1)^2 = x^2 + 2x + 1$$

In particular, we see that *composition is not commutative*, i.e.

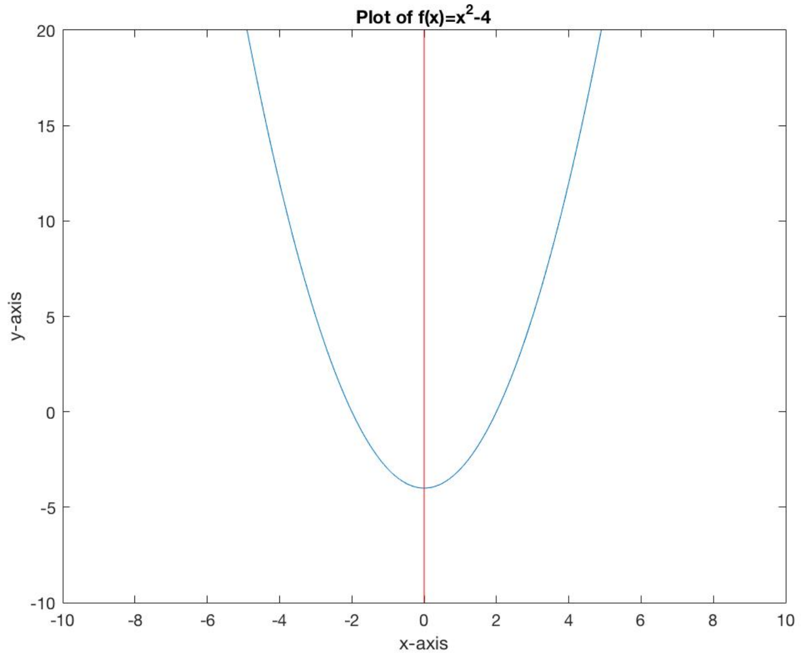
$$(f \circ g)(x) \neq (g \circ f)(x)$$

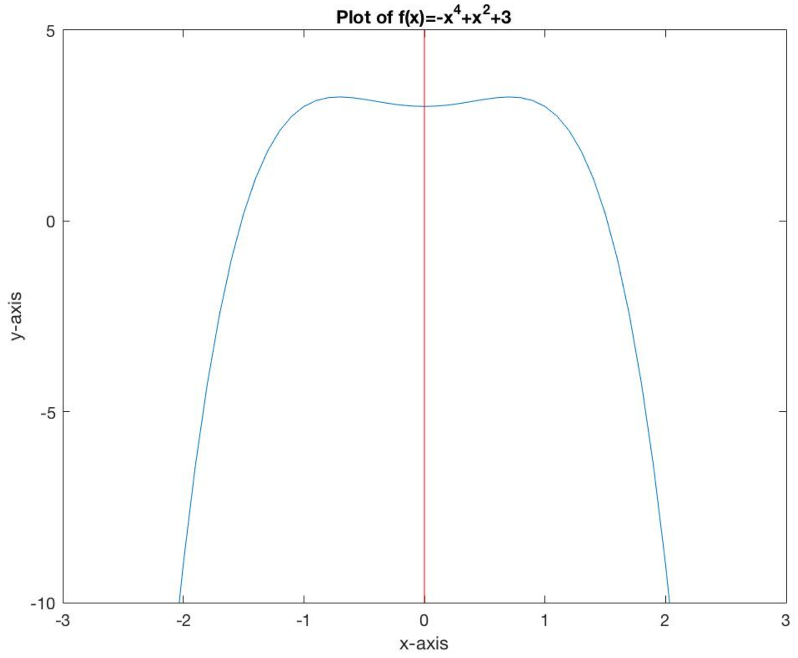
3.5.1 Plotting Functions

- **Drawing a function in the Cartesian plane is extremely useful in understand the relationship it defines.**
- **One can always attempt to plot a function by computing many pairs $(x, f(x))$, and plotting these on the Cartesian plane.**
- **However, simpler qualitative observations may be more efficient. We will discuss of a few of these notions before moving on to some standard function plots to know.**

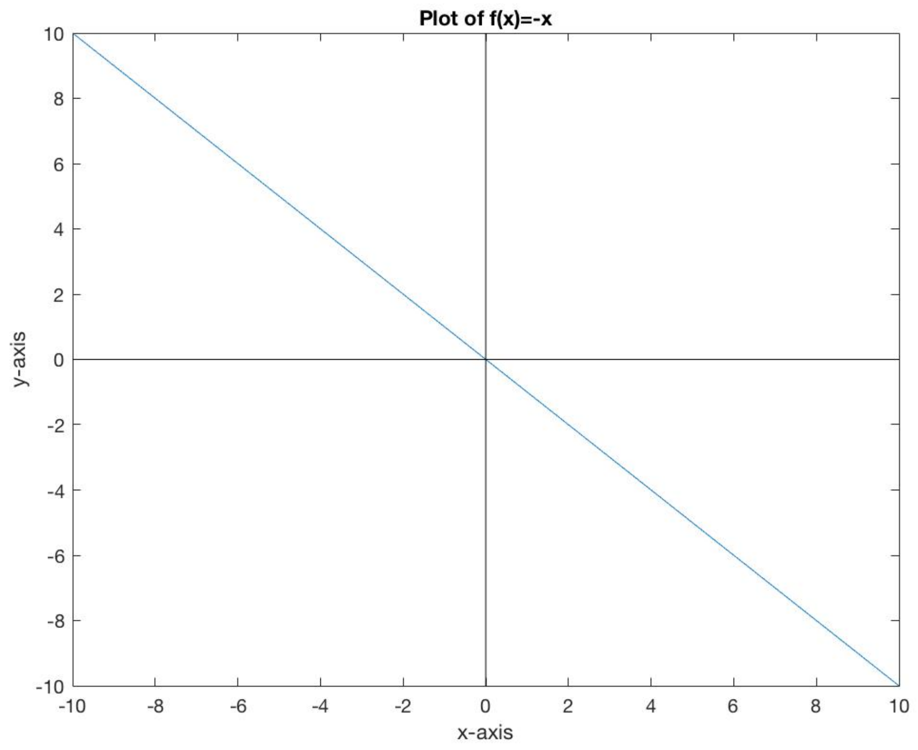
3.5.2 Symmetry of Functions

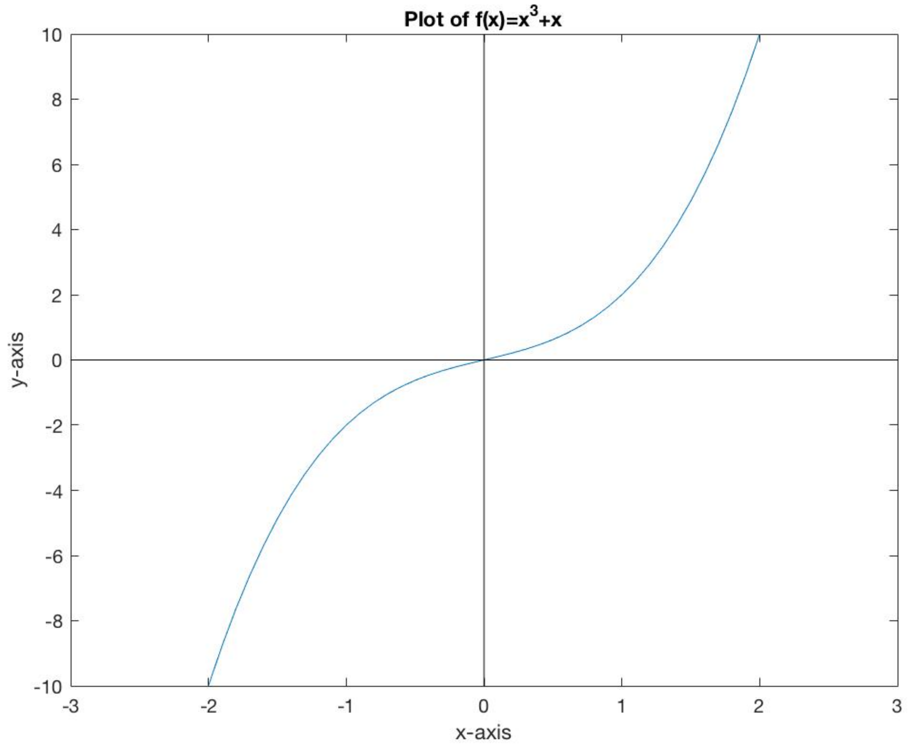
- **A function $f(x)$ is said to be *even/is symmetric about the y-axis* if for all values of $x, f(x) = f(-x)$.**
- **Functions that are even are mirror images of themselves across the y-axis.**





- A function $f(x)$ is said to be ***odd/has symmetry about the origin*** if for all values of x, $f(-x) = -f(x)$
- Functions that are odd can be reflected over the x-axis, then the y-axis.





3.5.3 Transformation of Functions

It is also convenient to consider some standard transformations for functions, and how they manifest visually:

- $f(x) \mapsto f(x+a)$ **shifts the function to the left by** a **if** a **is positive, and to the right by** a **if** a **is negative.**
- $f(x) \mapsto f(x)+b$ **shifts the function up by** b **if** b **is positive, and down by** b **if** b **is negative.**
- $f(x) \mapsto f(-x)$ **reflects the function over the** y **-axis.**
- $f(x) \mapsto -f(x)$ **reflects the function over the** x **-axis.**

3.6.0 Inverse Functions

Let $f(x)$ be a function. The *inverse function f* is the function that "undoes" $f(x)$; it is denoted $f^{-1}(x)$.

More precisely, for all x in the domain of $f(x)$,

$$(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$$

3.6.1 Remarks on Inverse Functions

- **Not all functions have inverse functions; we will show how to check this shortly.**
- **Note that $f^{-1}(x) \neq (f(x))^{-1}$, that is, inverse functions are not the same as the reciprocal of a function. The notation is subtle.**
- **The domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$.**

3.6.2 Horizontal Line Test

- **Recall that one can check if a plot in the Cartesian plane is the plot of a function via the *vertical line test.***
- **One can check whether a function $f(x)$ has an inverse function via the *horizontal line test*: the function has an inverse if every horizontal line intersects the plot of $f(x)$ at most once.**