5. Geometry

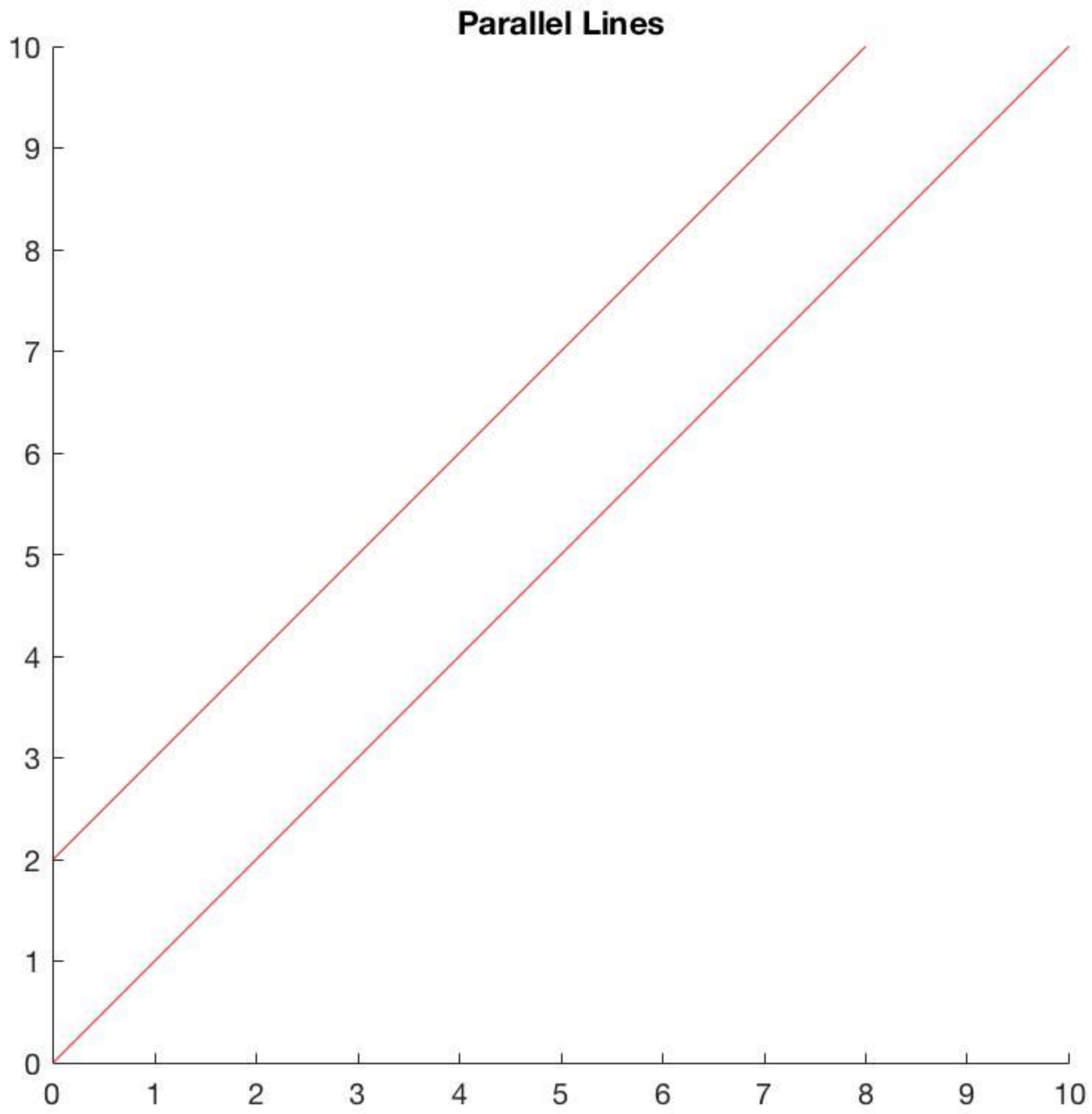
5.1 Lines

- **The study of planar geometry goes back at least to Euclid (~300 BCE).**

- **Lines are an important part of this theory.**

- **Two lines are said to be *parallel* if they never intersect, or equivalently, if they have the same slope.**

- **Two lines are said to be *perpendicular* if they intersect at a right angle.**


Parallel Lines
0
1
2
3
4
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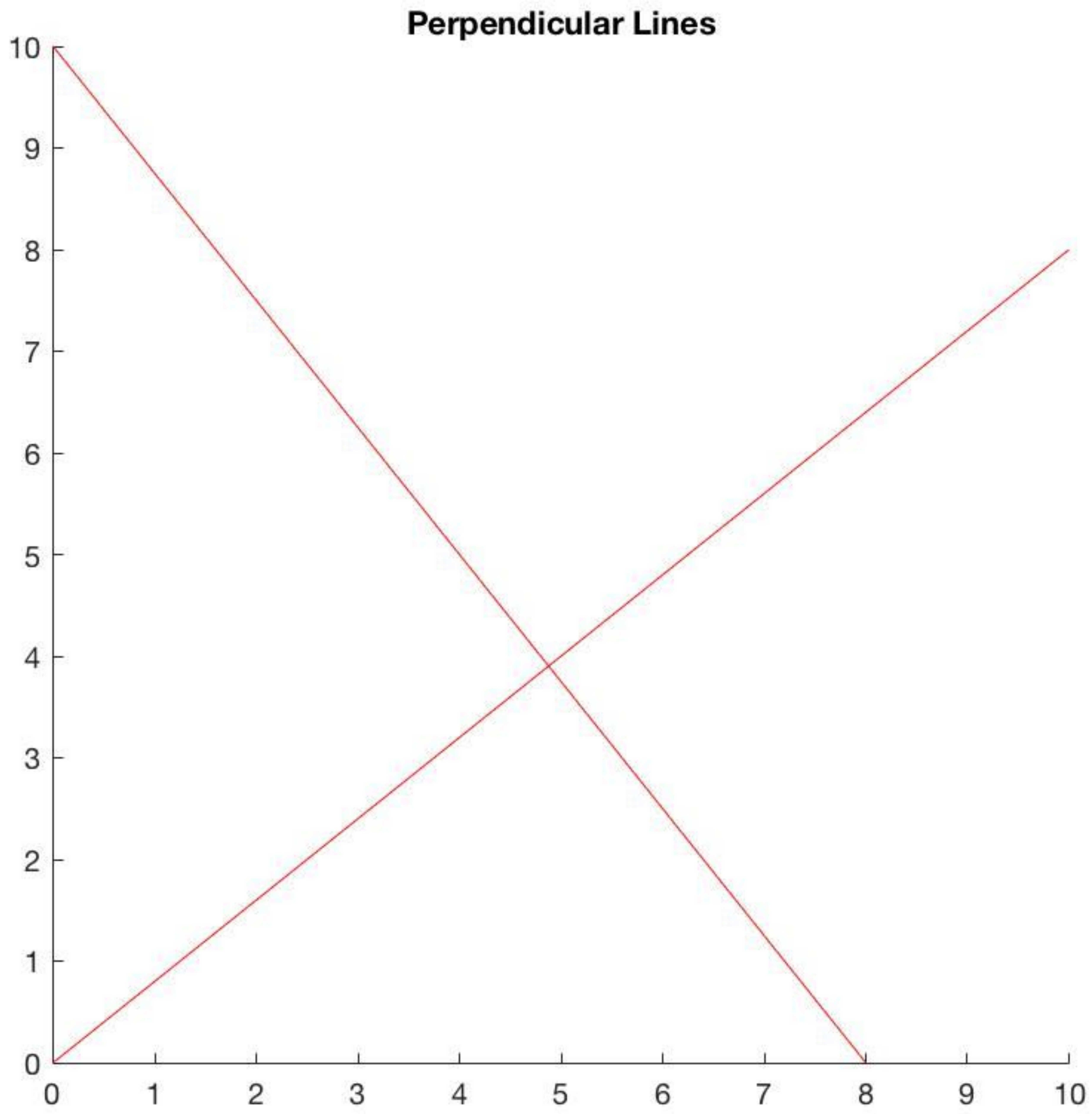
10


Perpendicular Lines
0
1
2
3
4
5
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7
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10
0
1
2
3
4
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6
7
8
9
10

Analysis of Slopes

- Given formulae for two lines, one can quickly determine if they are parallel or perpendicular by analyzing their slopes:

$$
\begin{aligned}
y =& m_1 x + b_1 \\
y =& m_2 x + b_2
\end{aligned}
$$

- The lines are *parallel* if $m_1 = m_2$.

- The lines are perpendicular if $m_1 = -\dfrac{1}{m_2}$.

Determine if the following lines are parallel, perpendicular, or neither:

- $y=3x+1$, $y=3x-6$
- $y=3x+1$, $y=1/3x-2$
- $y=x+1$, $y=-x+2$
- $3y=4-x$, $2x=6-6y$

5.2.1 General Triangles

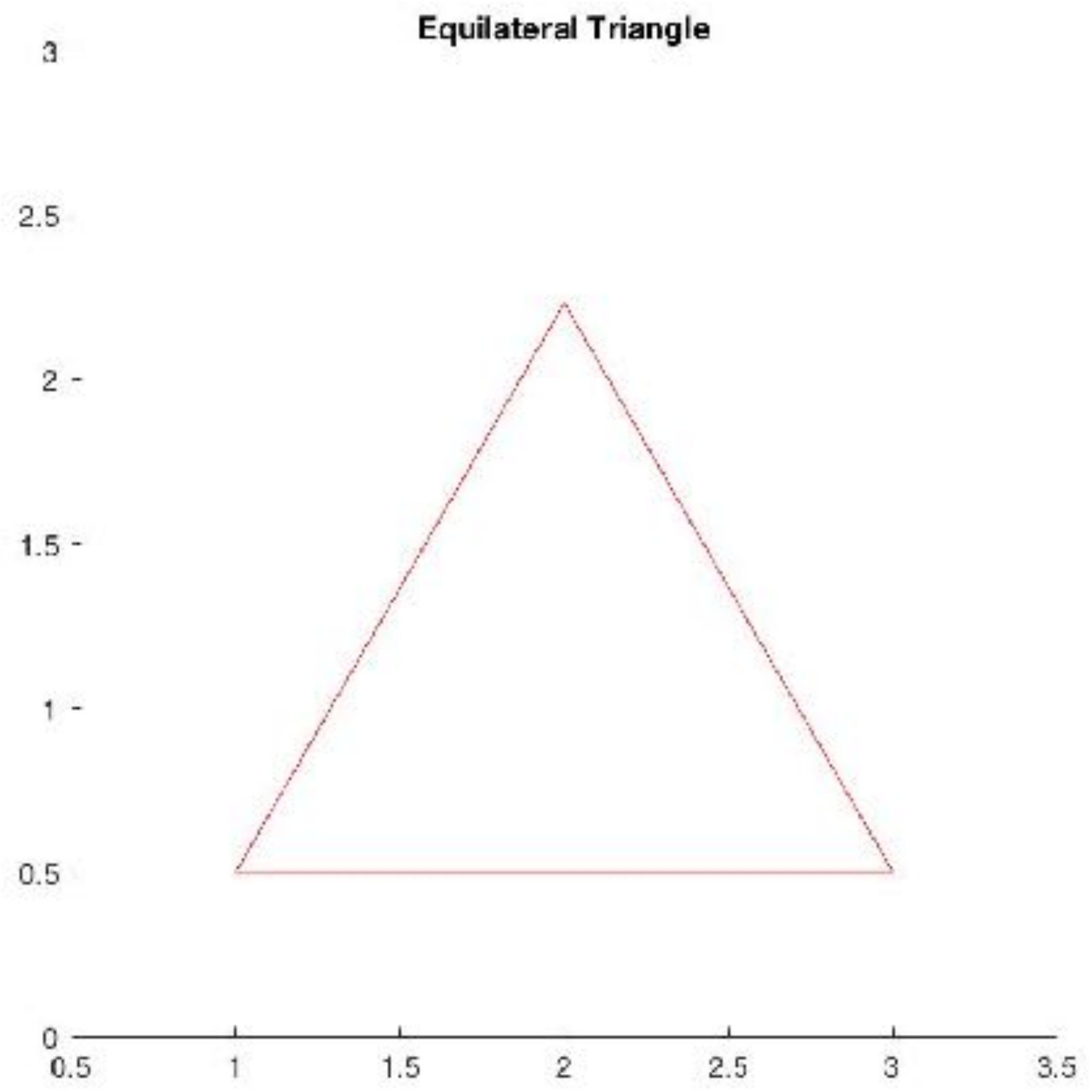
- **Triangles are among the simplest *polygons*.**

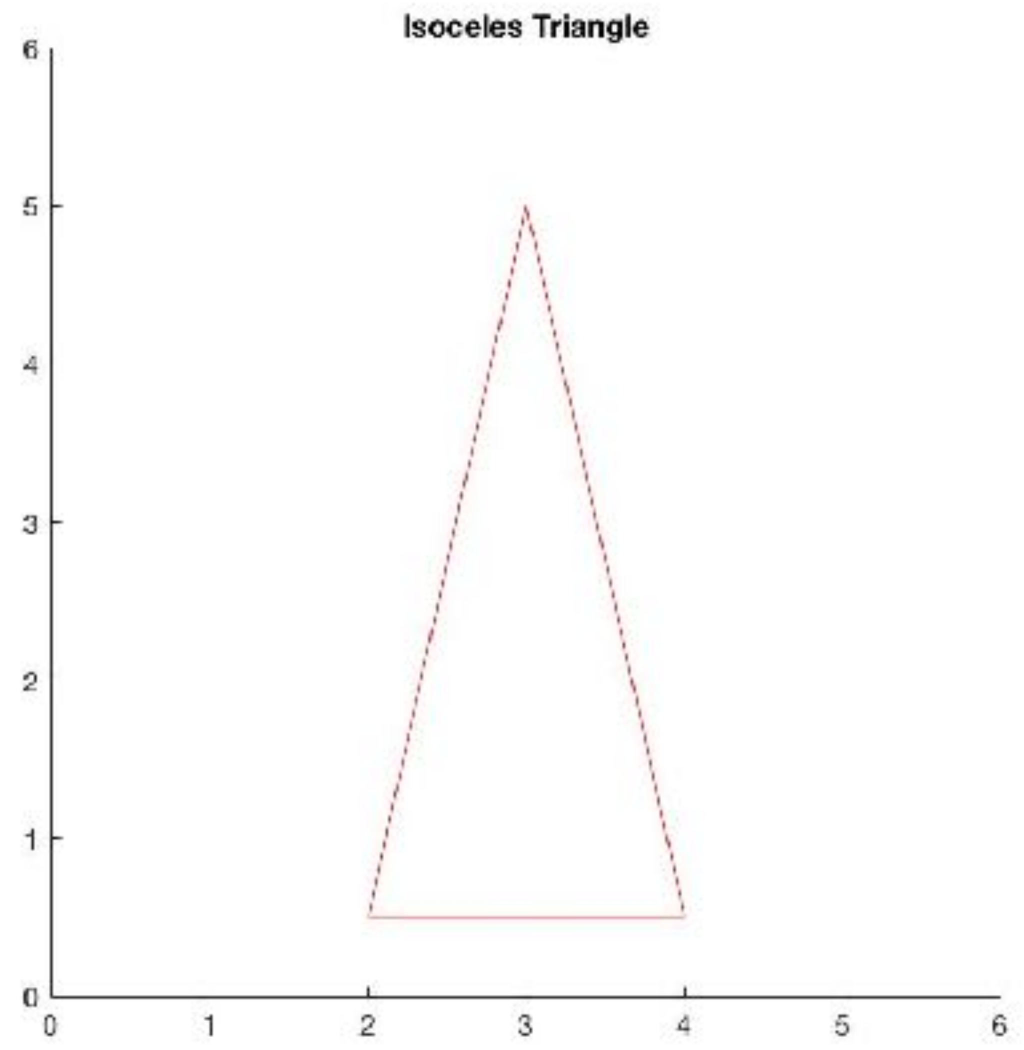
- **They are polygons with the fewest possible number of sides: 3.**

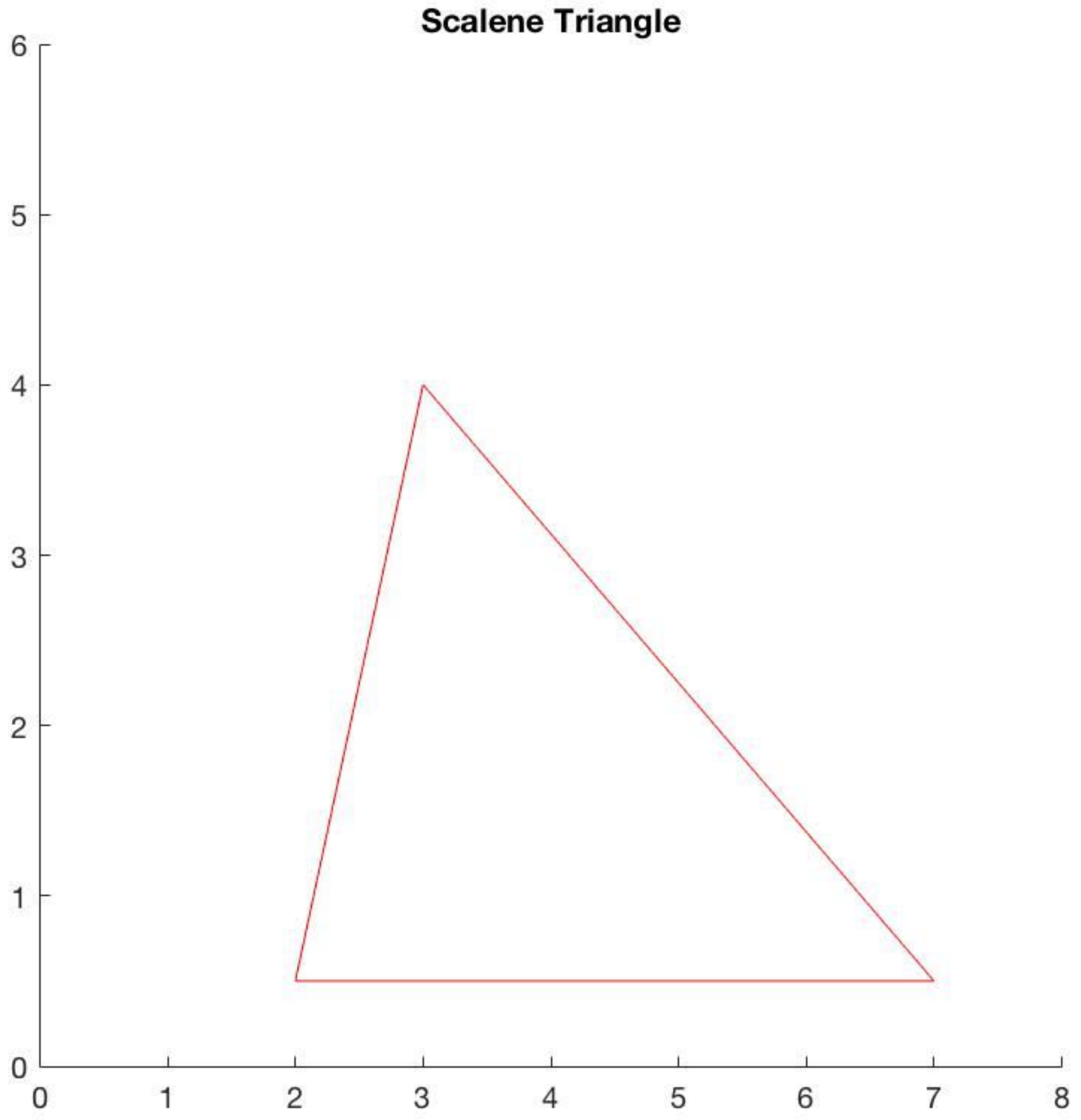
- **Triangles can be classified based on their angles or side lengths (it is equivalent to discuss one or the other!)**

- **Triangles also have several very nice formulas governing in them, particularly if one of the angles of the triangle is equal to 90°.**

Triangle classification by side length






Scalene Triangle
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

Area and Perimeter of Triangles

- The *perimeter* of a polygon is the length of all its sides.
- The *area* of a polygon is the amount of space it encloses.
- For a triangle with *base length* B and *height length* H, the area of the triangle may be computed as:

$$Area = \frac{1}{2} B \times H$$

Find the areas of the following triangles:

- B=7, H=4
- B=8, H=4 (right triangle)
- B=2, H=2

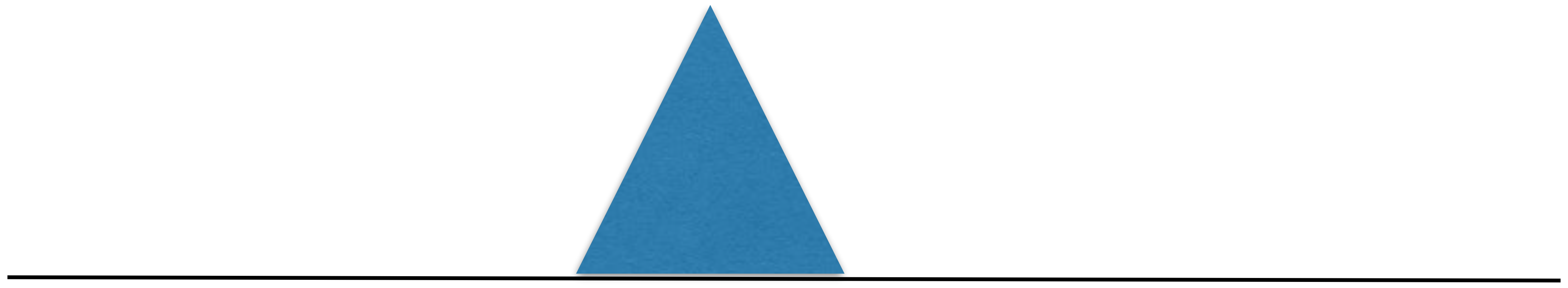
Regarding Angles

- **The angles of a triangle must sum to 180°.**
- **An equilateral triangle has all angles of equal size. They are thus all of size $\dfrac{180}{3} = 60^\circ$.**
- **An *angular bisector* is a line that splits an angle into two equal parts.**

Find the missing angles:

- 105, 40, ?

- 30, missing angles are opposite to equal side lengths (so use isosceles trick),
- Top angle=50, sides are same, want exterior angle.

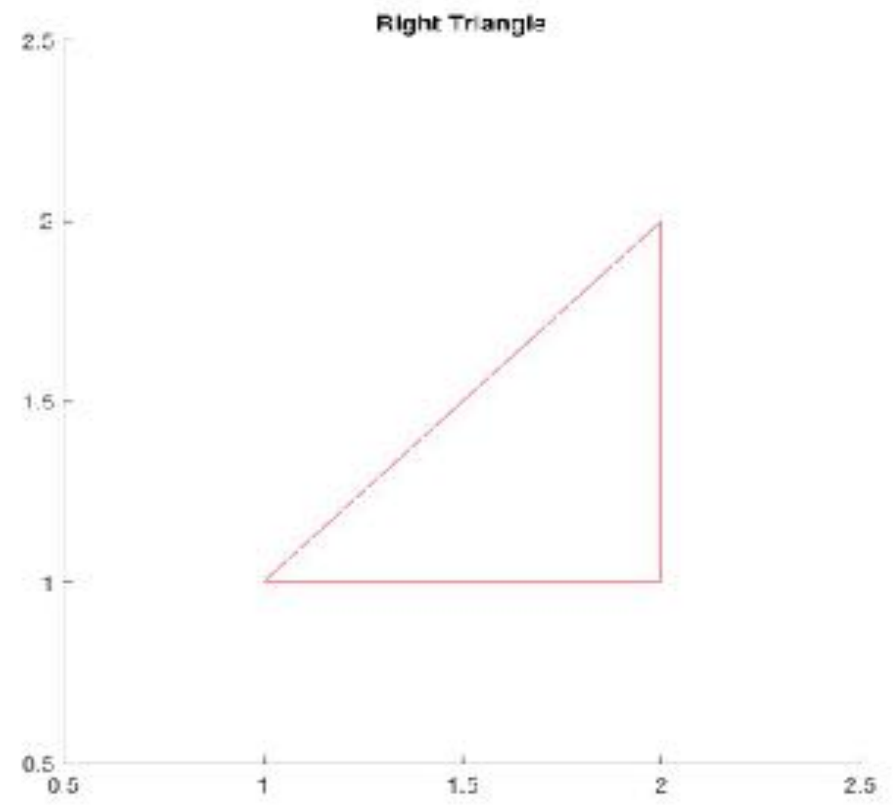


5.2.2 Right Triangles

Pythagorean Theorem

- A triangle with one angle of size 90° is called a *right triangle.*

- Right triangles have a special rule governing the length of their sides, given as the famous Pythagorean theorem:



$$a^2 + b^2 = c^2$$

Find the lengths of the missing sides:

- x,5,8
- x,4,5
- 5,7,x

Find the areas of the following triangles:

- B=3, Hyp=8, so H=sqrt(55)
- B=5, Hyp=7, so H=sqrt(24)

5.3.1 Quadrilaterals I

- **One can consider polygons besides triangles. Those with four sides are called *quadrilaterals.* Those with higher numbers of sides have specialized names, often with Greek roots.**

- **We will consider area formulae for special quadrilaterals: parallelograms, including rectangles, rhombi, squares, and a more general class of quadrilaterals called trapezoids.**

Parallelograms

- **Parallelograms are quadrilaterals with opposite sides parallel.**

- **Parallelograms include rectangles, rhombi, and squares as special cases.**

- **Parallelograms have area given by their base multiplied by their height or *altitude.***

$$Area = B \times H$$

- **Parallelograms have the property that adjacent angles sum to 180°, and opposite angles are equal.**

Find the area of the following parallelograms:

- B=4, H=4
- B=10, H=10

Find the missing angles:

- parallelogram, adjacent angle=30
- parallelogram, opposite angle=30
- parallelogram, our angle is x+50, adjacent angle is 2x-40

Rectangles

- **Rectangles are parallelograms with all angles equal to 90°**

- **The length of diagonals of rectangles may be found using the Pythagorean Theorem:**

$$D = \sqrt{B^2 + H^2}$$

Rhombi

- **A *rhombus* is a parallelogram with all sides of equal length.**

Squares

- A square is a quadrilateral that is both a rectangle and a rhombus.

- It is the simplest quadrilateral, and also the most constrained in its definition.

5.3.2 Quadrilaterals II

Trapezoids

- A trapezoid is a quadrilateral that has one pair of parallel sides.
- So, parallelograms are trapezoids, but not all trapezoids are parallelograms.
- For a trapezoid, the area is found by accounting for both of the two parallel *bases:*

$$Area = \left(\frac{B_1 + B_2}{2}\right) \times H$$

Find the areas of the following quadrilaterals:

- Trap, B1=8, B2=6, H=7
- Trap, B1=10, B2=15, H=4

Similar Polygons

- Two polygons are said to be similar if their angles are the same, in the same order.

- Equivalently, polygons are similar if, after a rotation, they are the same up to scaling.

- Similar polygons look like magnified of shrunk versions of each other.

- One can use this scaling ration to relate the areas, perimeters, diagonal lengths, side lengths, etc. of one polygon to the other.

5.4 Circles

- **Circles are sets of points that are at fixed distance to a *center* point.**

- **The distance from points on the circle to this center is called the *radius* of the circle.**

- **Circles do not fit into the polygon regime, because circles do not have edges per se.**

- **They may be though of as having infinitely many edges in a certain sense, which can be made precise with calculus.**

Area and Circumference of Circles

- The area of a circle is given in terms of its radius:

$$Area = \pi r^2$$

- The length of the circle is typically called *circumference* rather than perimeter, and may be computed as

$$C = 2\pi r$$

- One may also easily discuss the circumference in terms of *diameter* of the circle. The diameter is the length of a line going across the circle and through the center.

- **Hence, the diameter of a circle has length equal to twice that of the radius:**

$$D = 2r$$

- **With this, we see that the circumference may also be computed in terms of diameter as**

$$C = \pi D$$

Find the area and circumference of the following circles:

- $r = 3$
- $d = 10$

Arcs in a Circle

- One can discuss inscribed angles in a circle, and the corresponding arc length they cut off.

- The size of the angle is proportional to the length of the arc:

$$\frac{\text{Length Arc}}{\text{C}} = \frac{\text{Angle}}{360}.$$

- A similar principle holds for wedge areas:

$$\frac{\text{Area Wedge}}{\text{Area Circle}} = \frac{\text{Angle}}{360}.$$

Find the area of the following circular arcs:

- **r=5, theta=90**
- **arc length =10, theta = 30**