1. Algebra and Functions

1.1.1 Equations and Inequalities

1.1.2 The Quadratic Formula

1.1.3 Exponentials and Logarithms

1.2 Introduction to Functions

1.3 Domain and Range

1.4.1 Graphing Functions

1.4.2 Transformations of Functions

1.4.3 Inverse Function

1.5 Linear and Exponential Growth

1.1.1 Equations and Inequalities

- **Polynomials: linear, quadratic, higher order**

  $$ax + b = 0, ax^2 + bx + c = 0$$

- **Exponential and logarithms**

  $$y = a^x, y = \log_a(x)$$

- **Absolute Value**

  $$y = |a|$$

- **Equations and inequalities are often solved with algebraic manipulations.**
- **Equations typically have *discrete points* as solutions; inequalities typically have *regions* as solutions.**
- **Certain rules may be useful for solving particular equations, like the quadratic formula.**

1.1.2 The Quadratic Formula

Quadratic Formula

A formulaic approach to solving quadratic equations is the *quadratic formula:*

$$0 = ax^2 + bx + c \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, quadratic equations have two distinct roots, unless $b^2 - 4ac = 0$ **.**

Solve $x^2 - 7x + 9 = 0$

Solve $2x^2 + x - 3 = 0$

1.1.3 Exponents and Logarithms

Properties of Exponents

Basic Rules:

- $a^{x+y} = a^x a^y$ **(same base, different exponents)**
- $a^x b^x = (ab)^x$ **(different base, same exponent)**
- $(a^x)^y = a^{xy}$ **(iterated exponents)**
- $x^0 = 1$ **for any value of x (convention)**

Properties of Logarithms

Logarithms enjoy certain algebraic properties, related to the exponential properties we have already studied.

- $\log_a(xy) = \log_a(x) + \log_a(y)$

  (logarithm of a product)
- $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

  (logarithm of a quotient)
- $\log_a(x^y) = y\log_a(x)$

  (logarithm of an exponential)
- $\log_a(1) = 0$ **(logarithm of 1 equals 0)**

  $\log_a(a) = 0$

Solve $3^{x^2-1} = 1$

Solve $2^{2x-3} = \dfrac{1}{4}$

Solve $\log_7(-x+1)=2$

Solve $\ln(2x + 2) = 0$

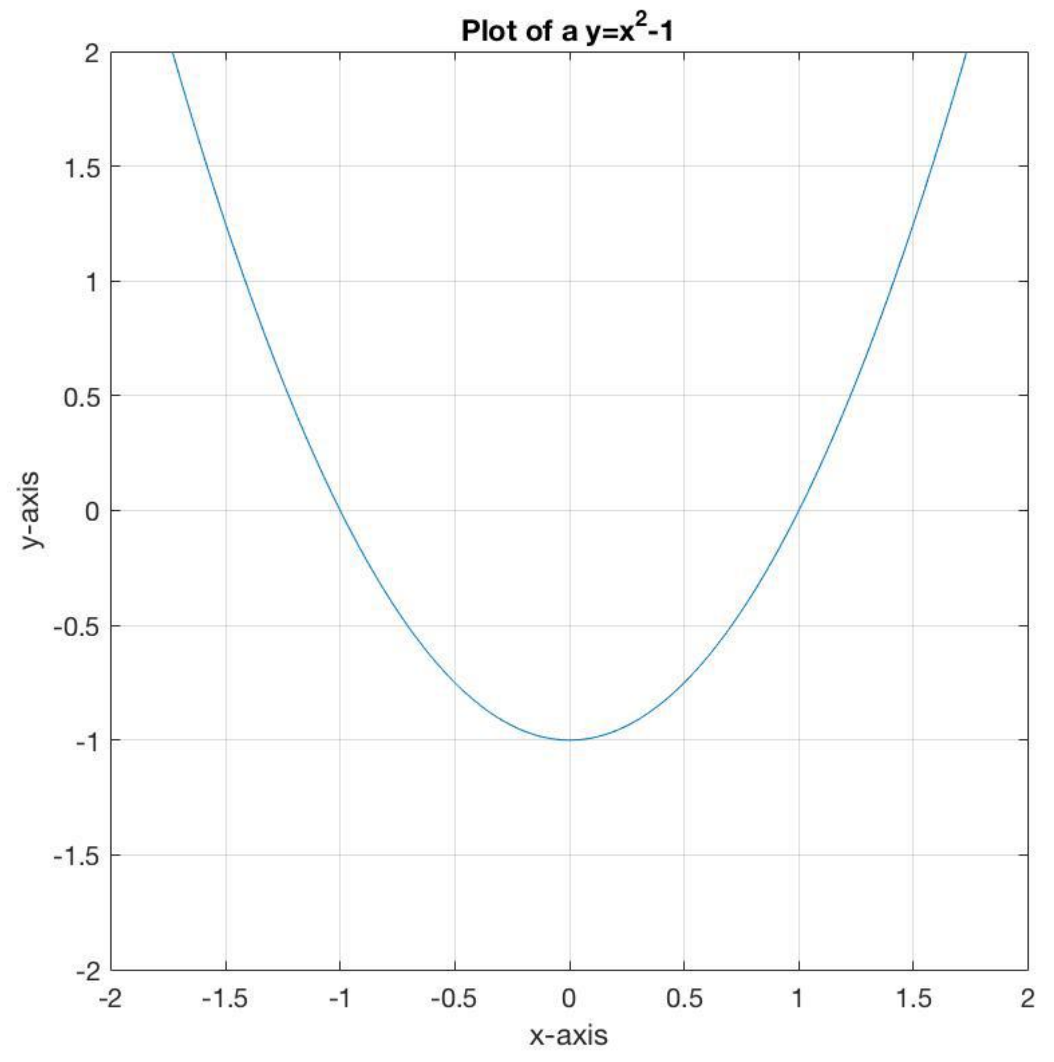
1.2 Introduction to Functions

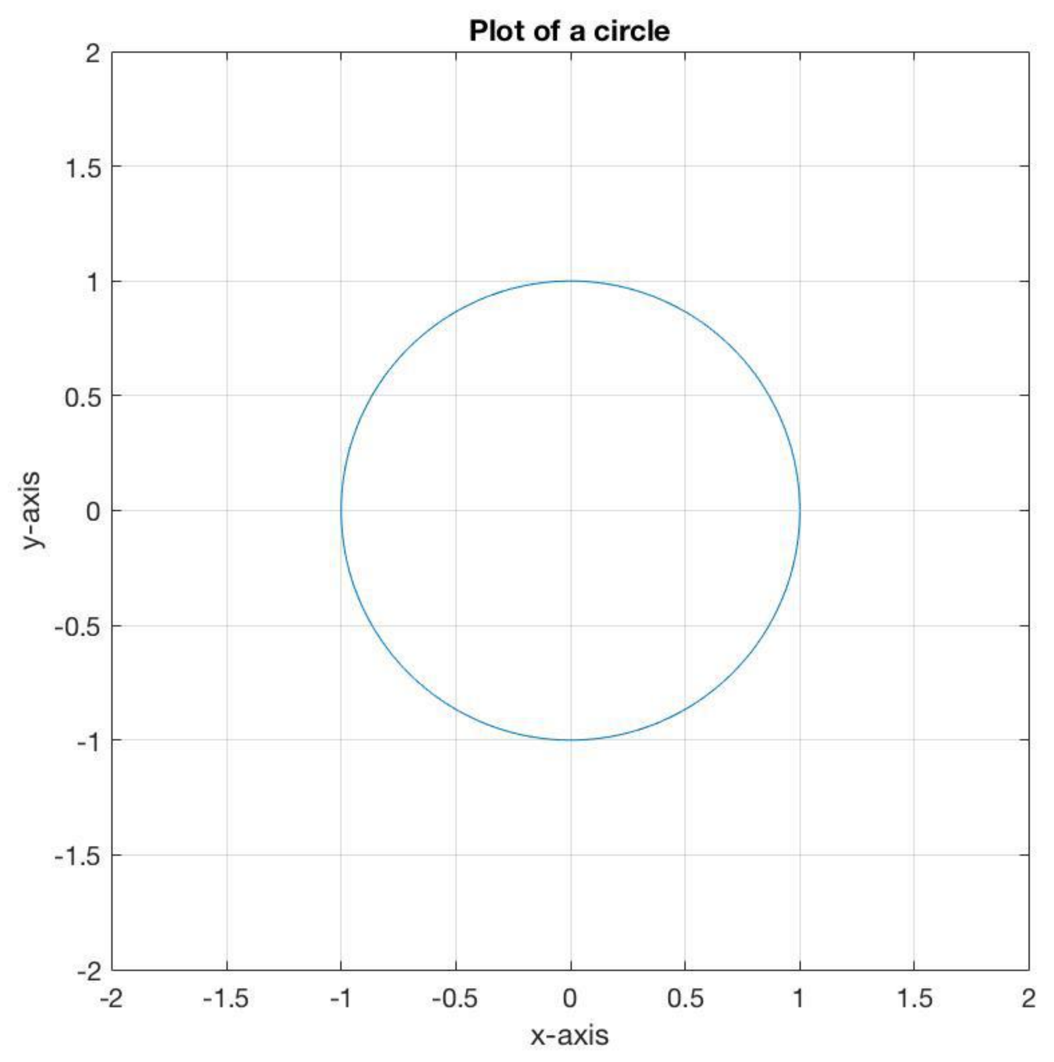
What is a Function?

- Functions are mathematical objects that send an input to a unique output.
- They are often, but not always, numerical.
- The classic notation is that $f(x)$ denotes the output of a function f at input value x.
- Functions are abstractions, but are very convenient for drawing mathematical relationships, and for analyzing these relationships.

Function or not?

One of the key properties of a function is that it **assigns a unique output to an input.**

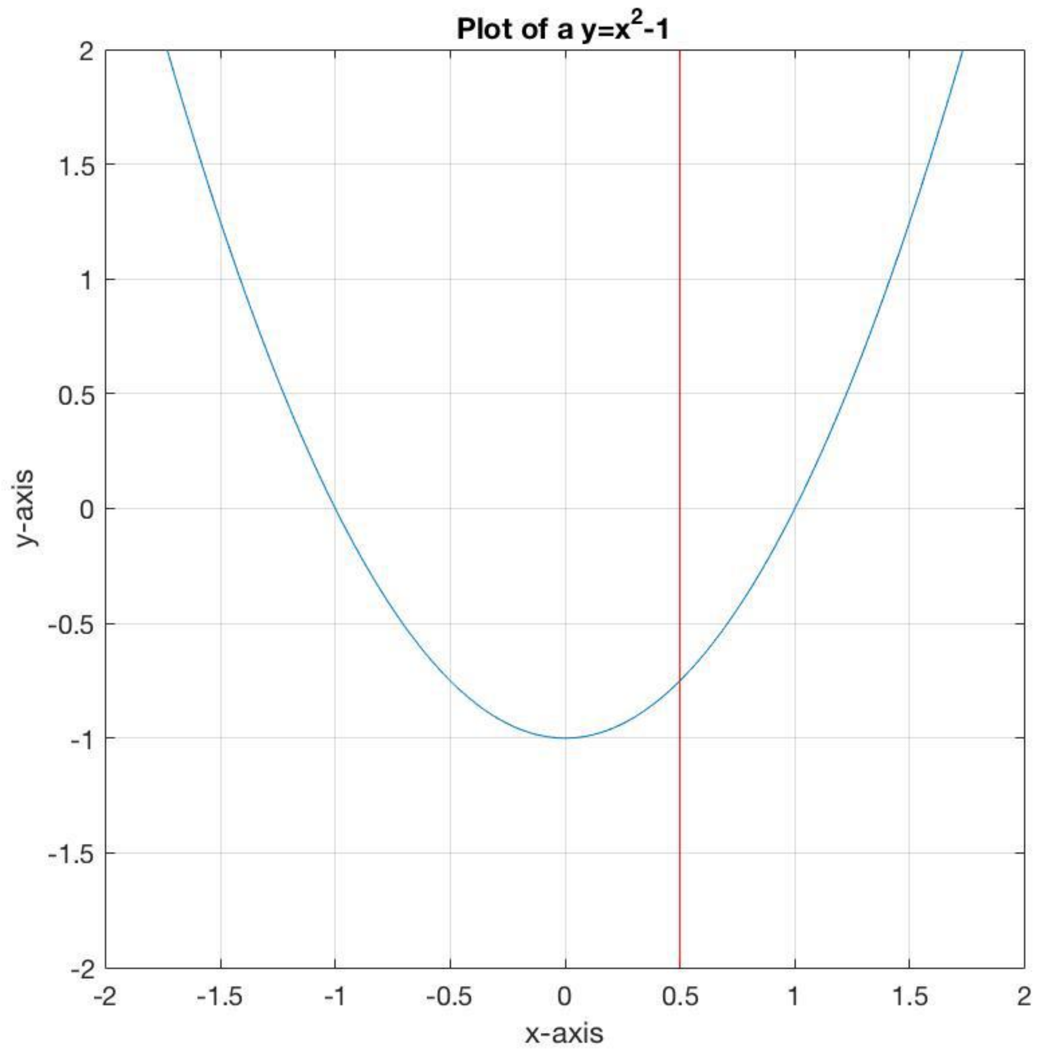


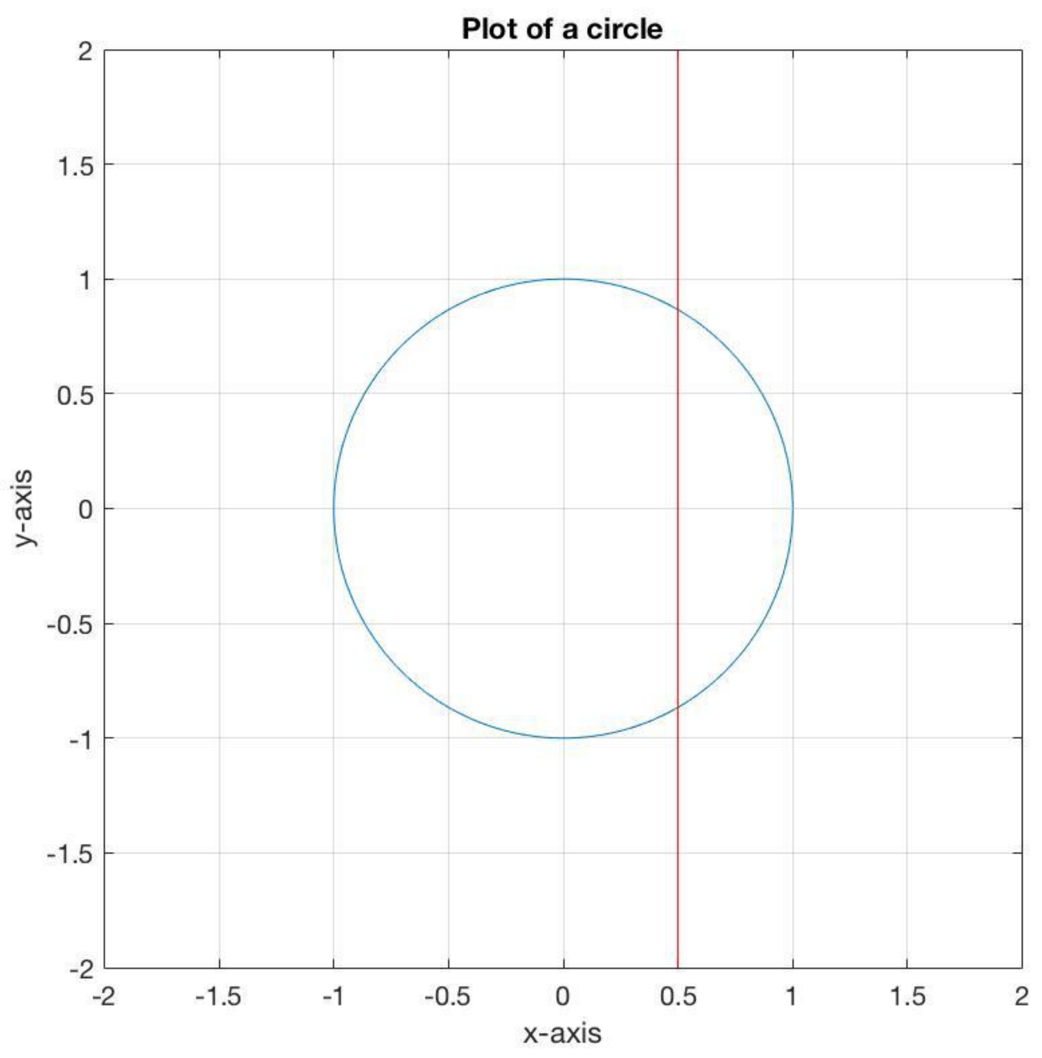


Vertical Line Test

A trick for checking if a mathematical relationship plotted in the Cartesian plane is a function is the *vertical line test.*

VLT: A plot is a function if and only if every vertical line intersects the plot in at most one place.





Plot $y = x^2 + 4$

Plot $x = y^2 + 4$

Function or not?

Representing with Functions

Functions are convenient for describing numerical relationships.

$$\text{Input} \xrightarrow{f} \text{Output}$$

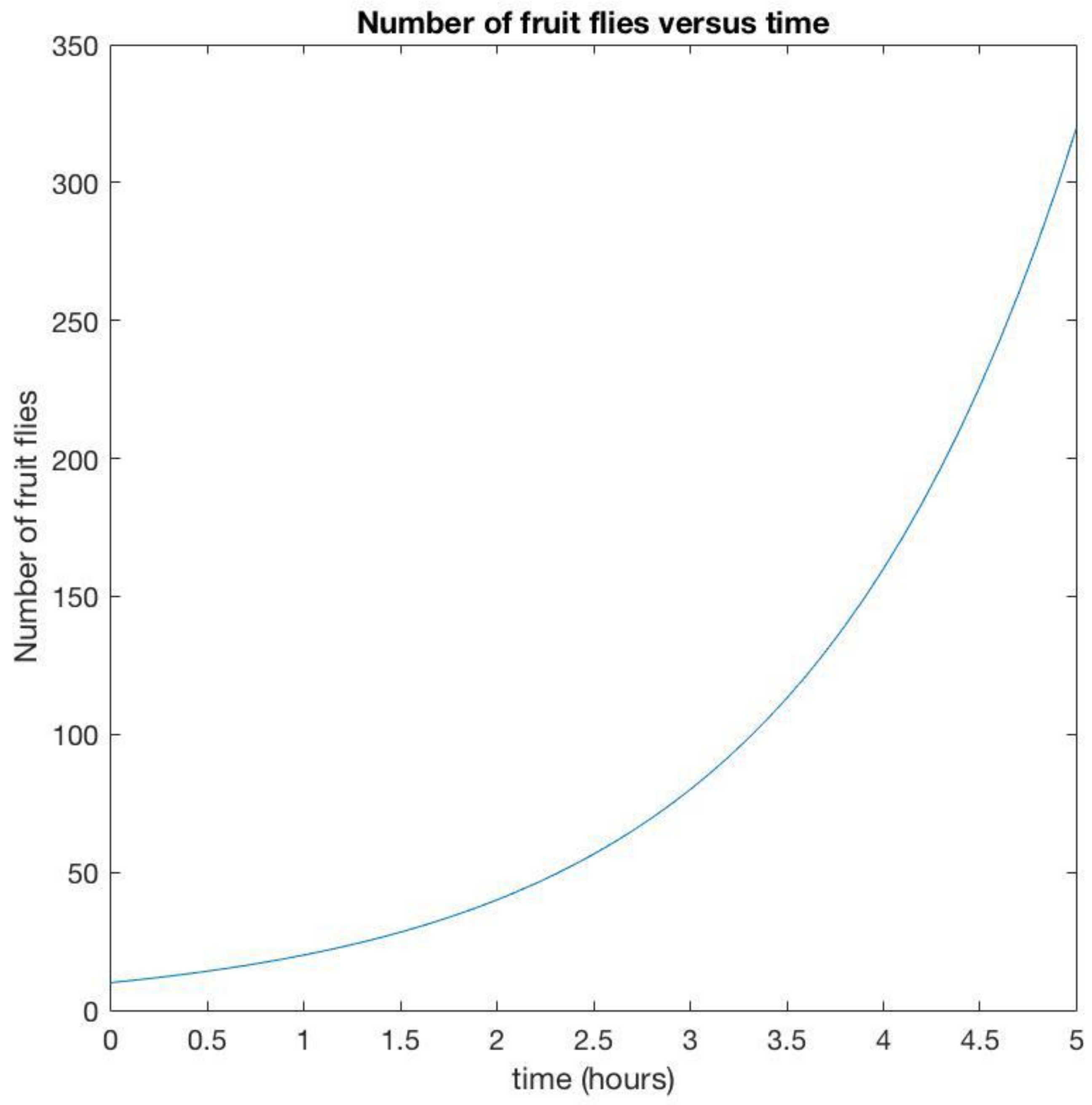
To model a relationship with functions, you simply need to understand how your input depends on your output.

Exponential Modeling

Exponential functions are more complicated than linear functions, but are very useful for things that, for example double in magnitude at a certain rate.

For example, suppose a colony of fruit flies starts with 10, and doubles every hour. Then the population of fruit flies at time t in hours is given as

$$P(t) = 10 \cdot 2^t$$



1.3 Domain and Range

Domain and Range of a Function

Let $f(x)$ be a function.

- **The *domain* of $f(x)$ is the set of allowable inputs.**
- **The *range* of $f(x)$ is the set of possible outputs for the function.**
- **These can depend on the relationship the functions are modeling, or be intrinsic to the mathematical function itself.**
- **They can also be inferred from the plot of $f(x)$, if it is available.**

Intrinsic Domain Limitations

Some mathematical objects have intrinsic limitations on their domains and ranges. Classic examples include:

- $f(x) = x^2$ **has domain** $(-\infty, \infty)$**, range** $[0, \infty)$**.**
- $f(x) = \sqrt{x}$ **has domain** $[0, \infty)$**, range** $[0, \infty)$**.**
- $f(x) = \log(x)$ **has domain** $(0, \infty)$**, range** $(-\infty, \infty)$**.**
- $f(x) = a^x$ **has domain** $(-\infty, \infty)$ **, range** $(0, \infty)$**.**
- $f(x) = \frac{1}{x}$ **has domain and range** $(\infty, 0) \cup (0, \infty)$**.**

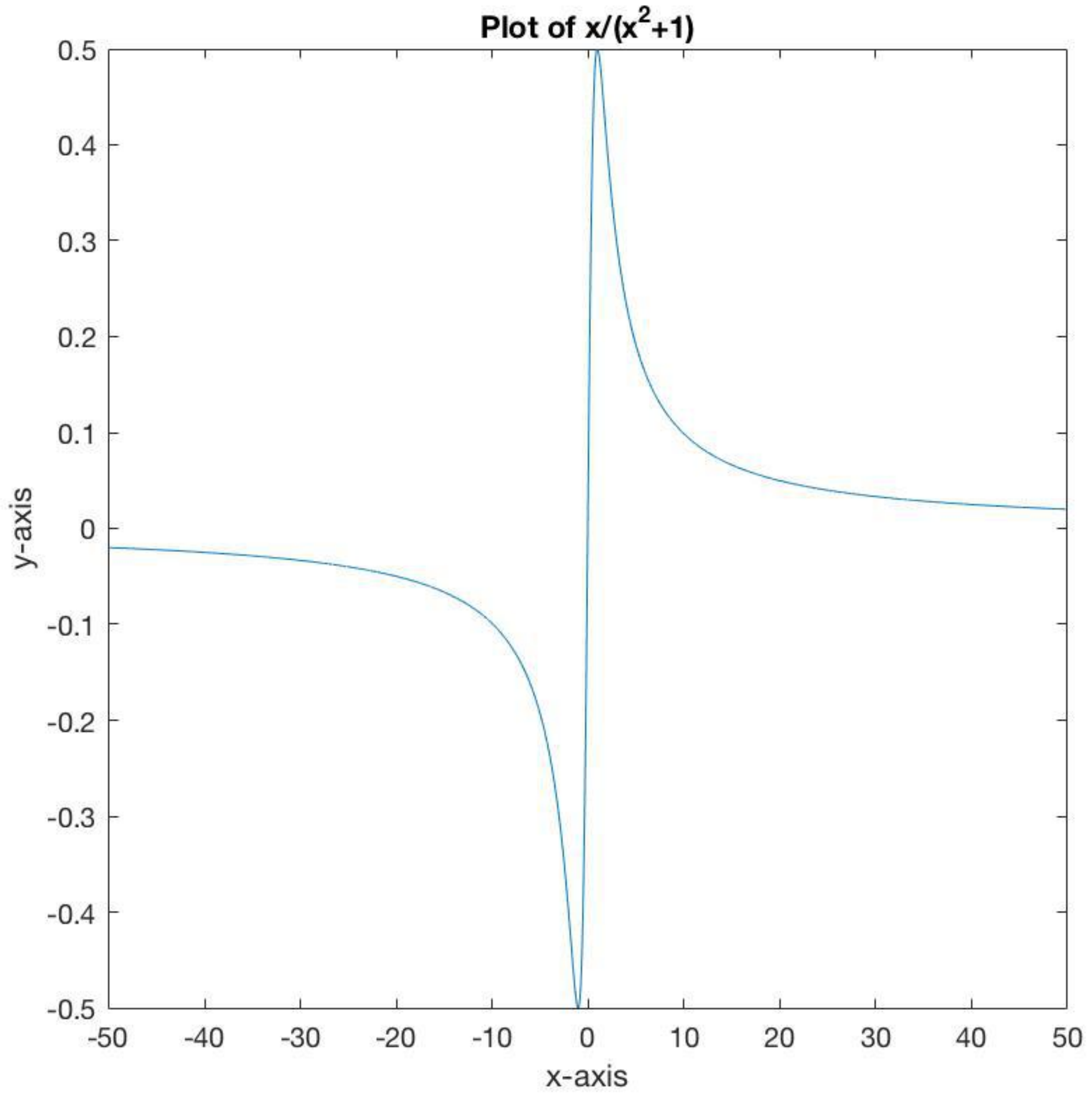
Visualizing Domain and Range

Given a plot of $f(x)$, one can observe the domain and range by considering what x and y values are achieved.

The function

$$f(x) = \frac{x}{x^2 + 1}$$

is hard to analyze, but its plot helps us guess its domain and range.



Find the domain and range of the following functions

$$f(x) = \sqrt{x+1}$$

$$g(x) = -\log_{10}(3x+2)$$

$$g(x) = e^{x+2}$$

$$f(x) = x^2 + 2$$

$$f(x) = -x - 7$$

$$h(x) = -|x| + 1$$

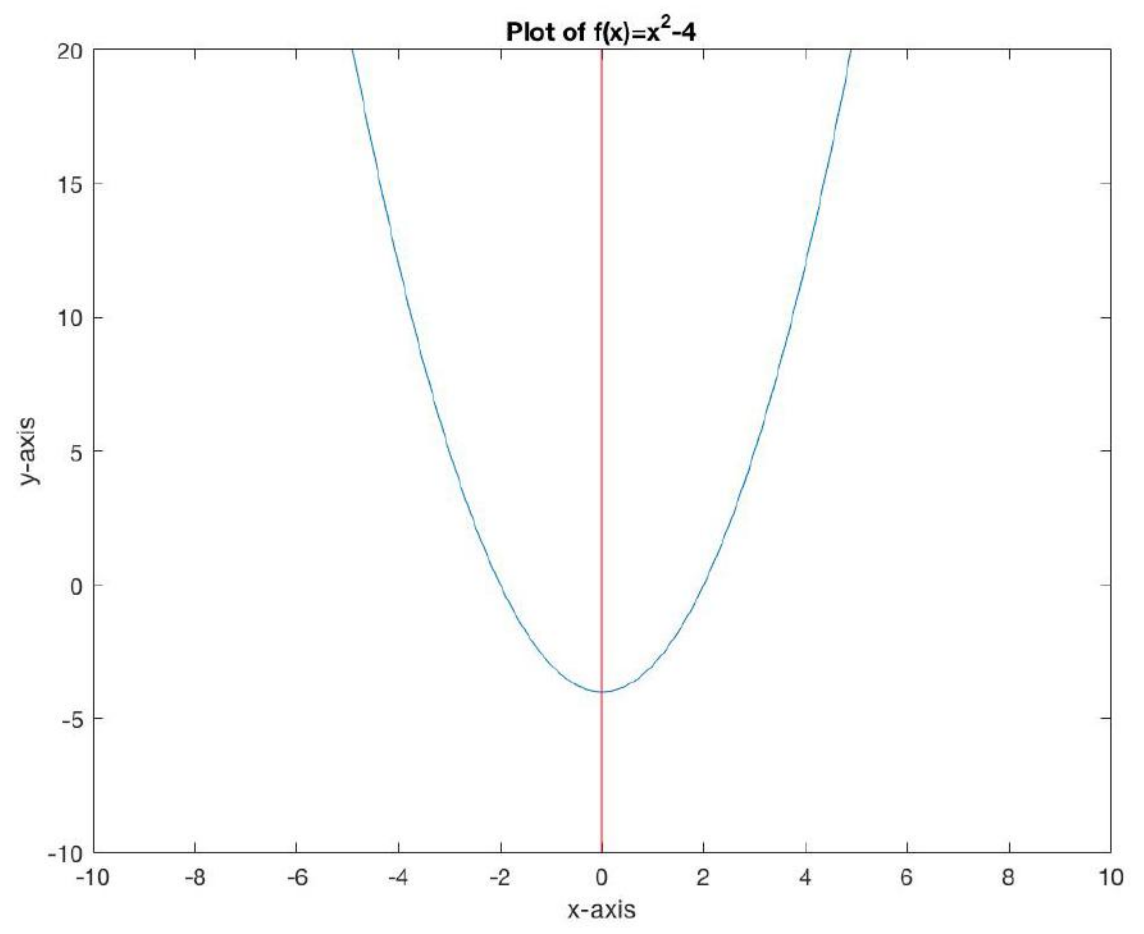
1.4.1 Graphing Functions

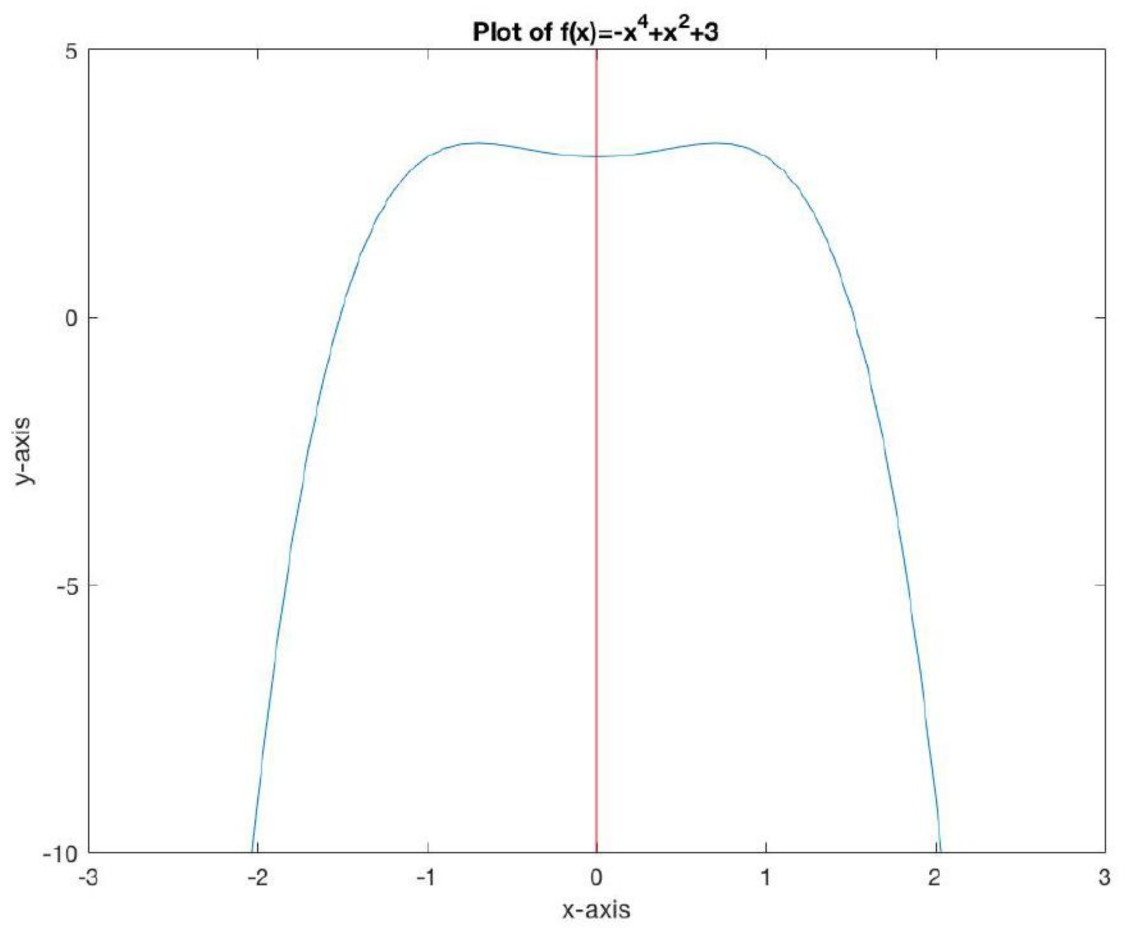
Plotting Functions

- **Drawing a function in the Cartesian plane is extremely useful in understand the relationship it defines.**
- **One can always attempt to plot a function by computing many pairs $(x, f(x))$, and plotting these on the Cartesian plane.**
- **However, simpler qualitative observations may be more efficient. We will discuss of a few of these notions before moving on to some standard function plots to know.**

Symmetry of Functions

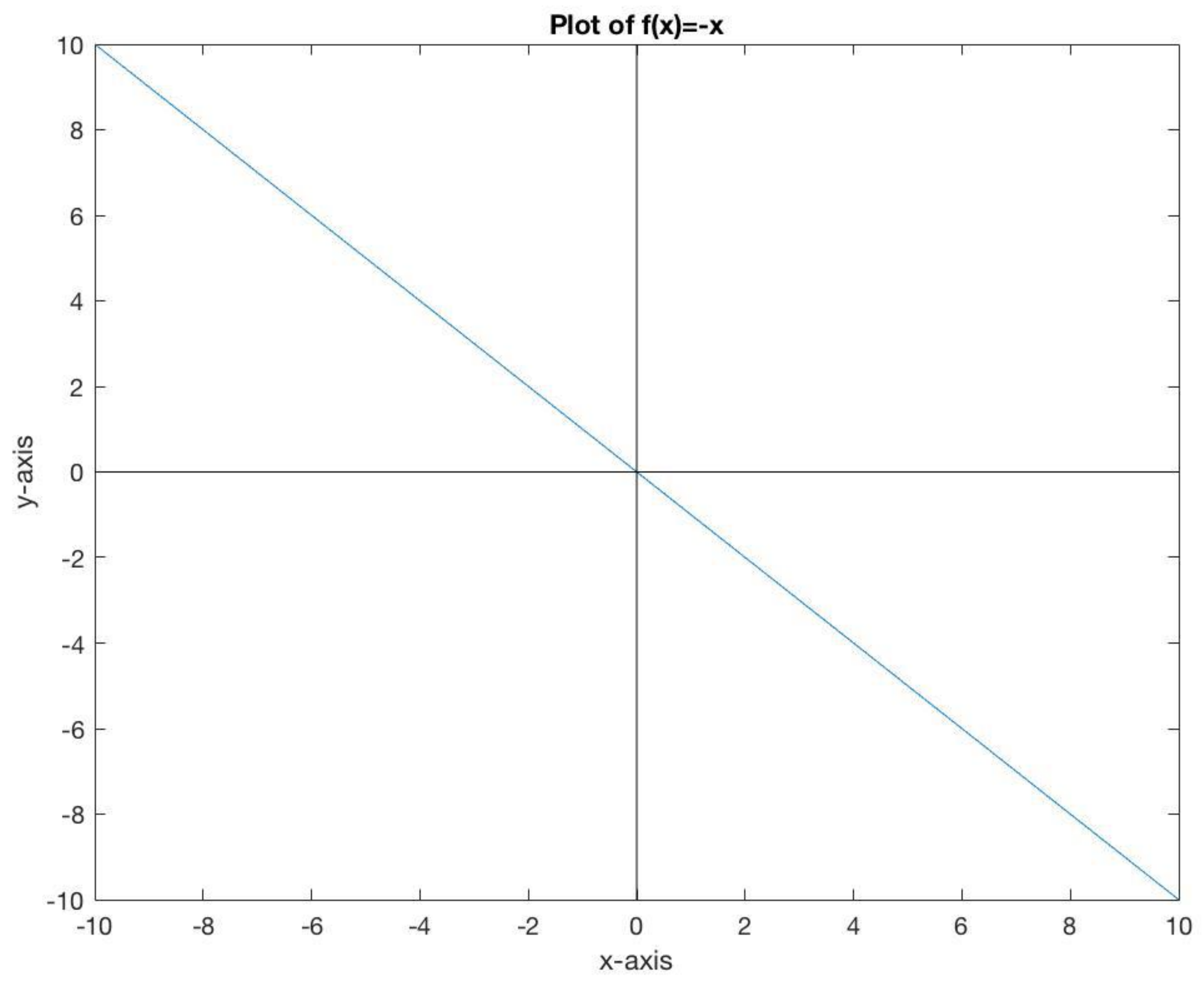
- A function $f(x)$ is said to be ***even/is symmetric about the y-axis*** if for all values of $x, f(x) = f(-x)$.
- Functions that are even are mirror images of themselves across the y -axis.

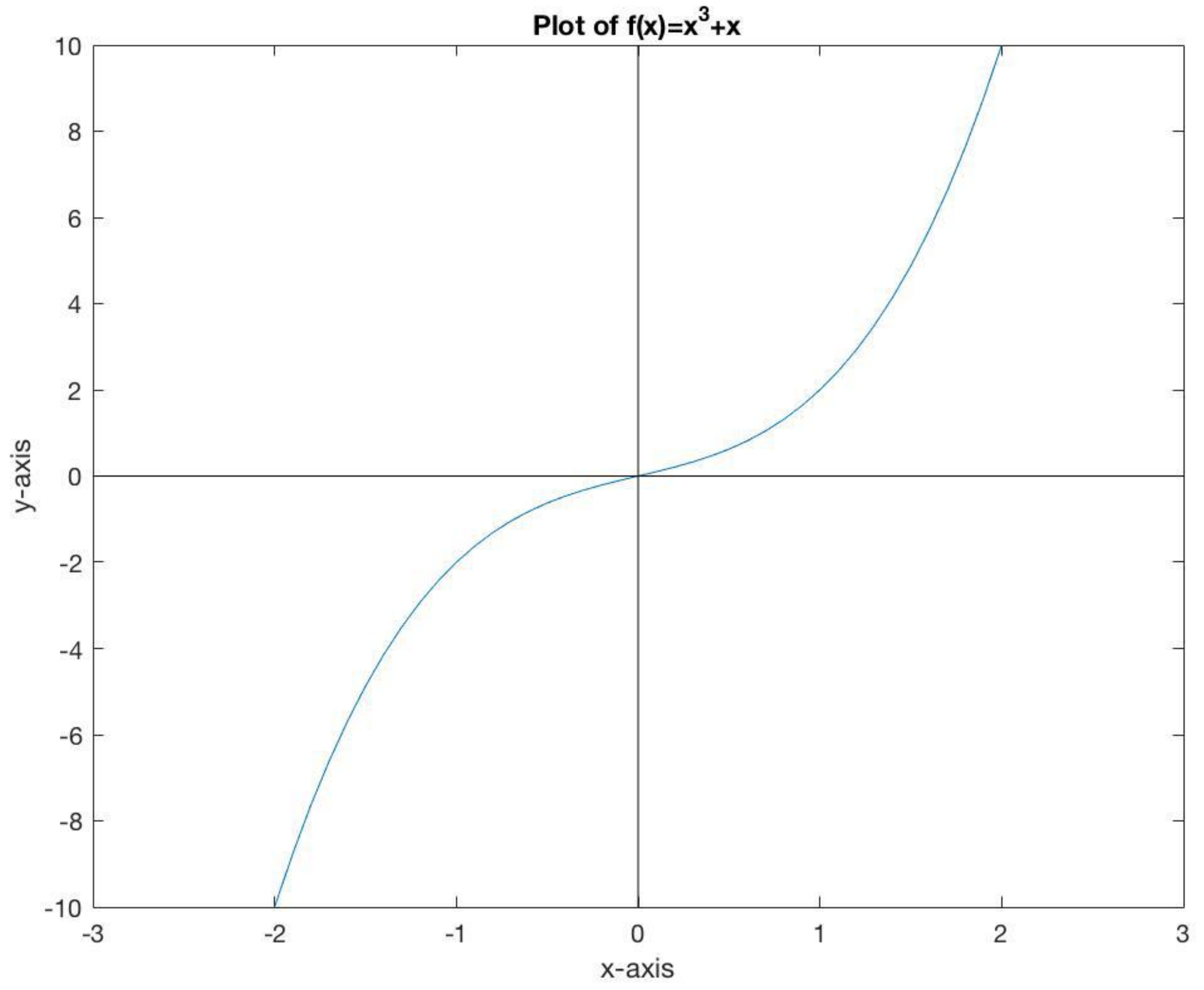




Symmetry of Functions

- A function $f(x)$ is said to be *odd/has symmetry about the origin* if for all values of x, $f(-x) = -f(x)$.
- Functions that are odd can be reflected over the x-axis, then the y-axis.





Are the following functions even, odd, or neither?

1.4.2 Transformation of Functions

Transformations of Functions

It is also convenient to consider some standard transformations for functions, and how they manifest visually:

- $f(x) \mapsto f(x+a)$ **shifts the function to the left by** a **if** a **is positive, and to the right by** a **if** a **is negative.**
- $f(x) \mapsto f(x) + b$ **shifts the function up by** b **if** b **is positive, and down by** b **if** b **is negative.**
- $f(x) \mapsto f(-x)$ **reflects the function over the** y**-axis.**
- $f(x) \mapsto -f(x)$ **reflects the function over the** x**-axis.**

Plot the following functions:

$$f(x) = 3x - 1$$

$$g(x) = -(x-1)^2$$

$$g(x) = e^{-x+1}$$

1.4.3 Inverse Functions

Inverse Functions

Let $f(x)$ be a function. The *inverse function f* is the function that "undoes" $f(x)$; it is denoted $f^{-1}(x)$.

More precisely, for all x in the domain of $f(x)$,

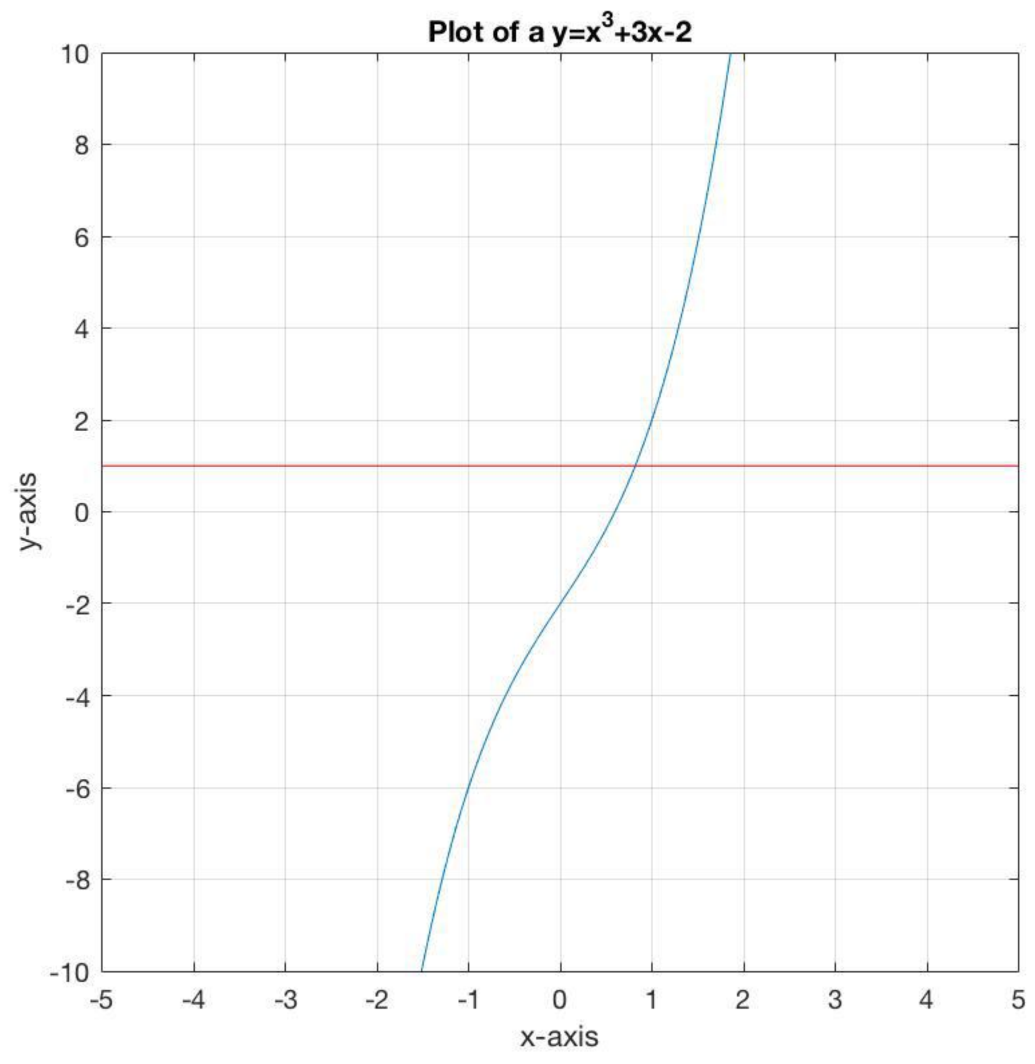
$$(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$$

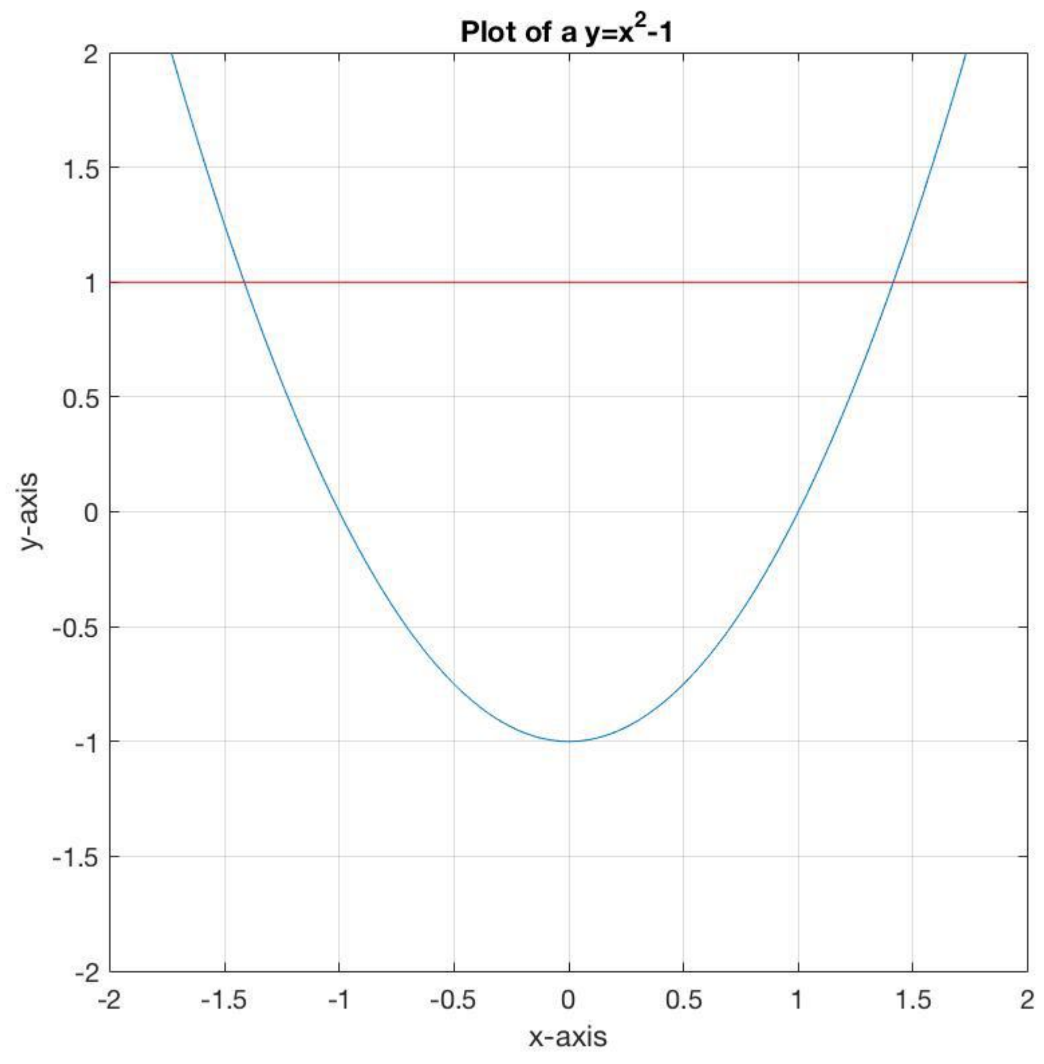
Remarks on Inverse Functions

- Not all functions have inverse functions; we will show how to check this shortly.
- Note that $f^{-1}(x) \neq (f(x))^{-1}$, that is, inverse functions are not the same as the reciprocal of a function. The notation is subtle.
- The domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$.
- Inverse functions can be plotted by taking the original function and reflecting across the line $y = x$

Horizontal Line Test

- **Recall that one can check if a plot in the Cartesian plane is the plot of a function via the *vertical line test.***
- **One can check whether a function $f(x)$ has an inverse function via the *horizontal line test*: the function has an inverse if every horizontal line intersects the plot of $f(x)$ at most once.**





For each of the following functions, determine if it has an inverse function on its range. If so, find it.

$$f(x) = x^2 + 1$$

$$f(x) = \log(x - 2)$$

$$g(x) = -2x - 1$$

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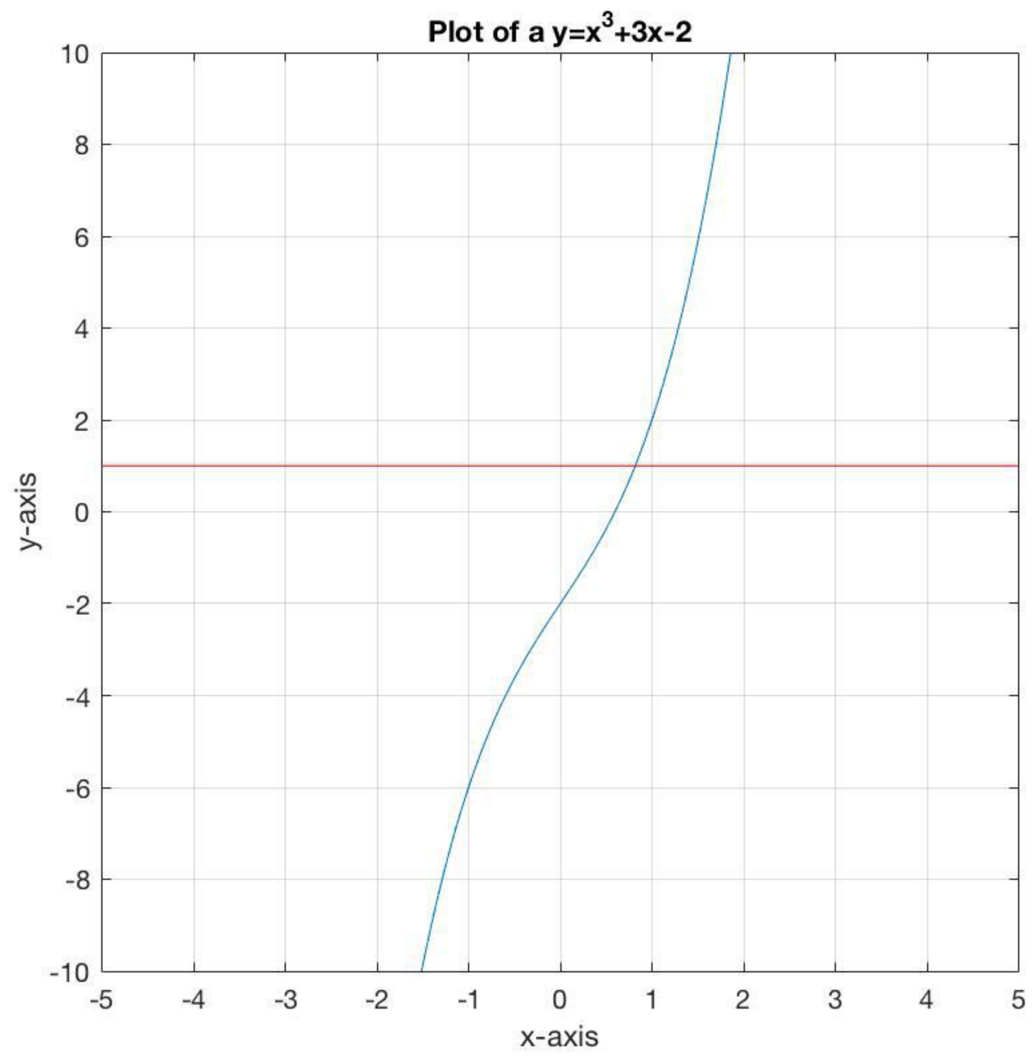

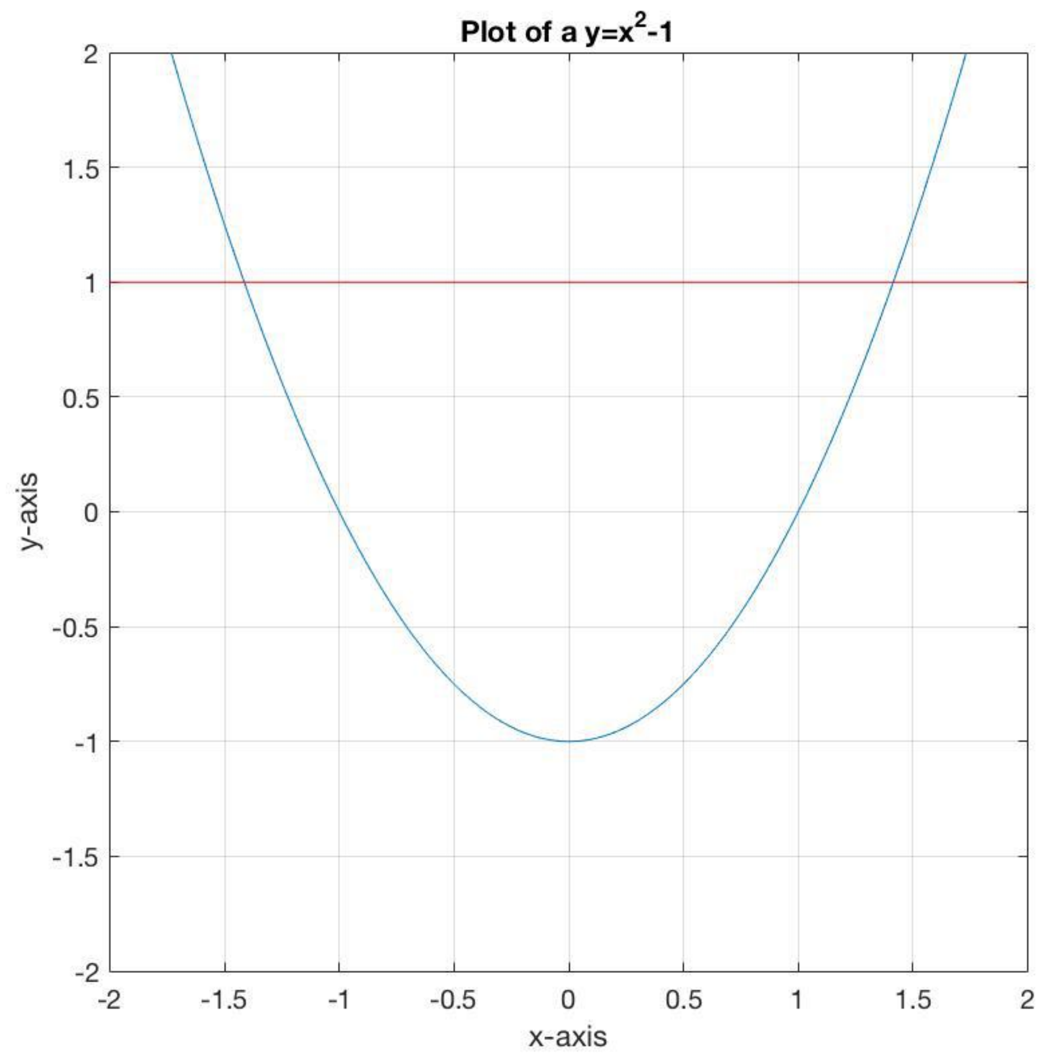

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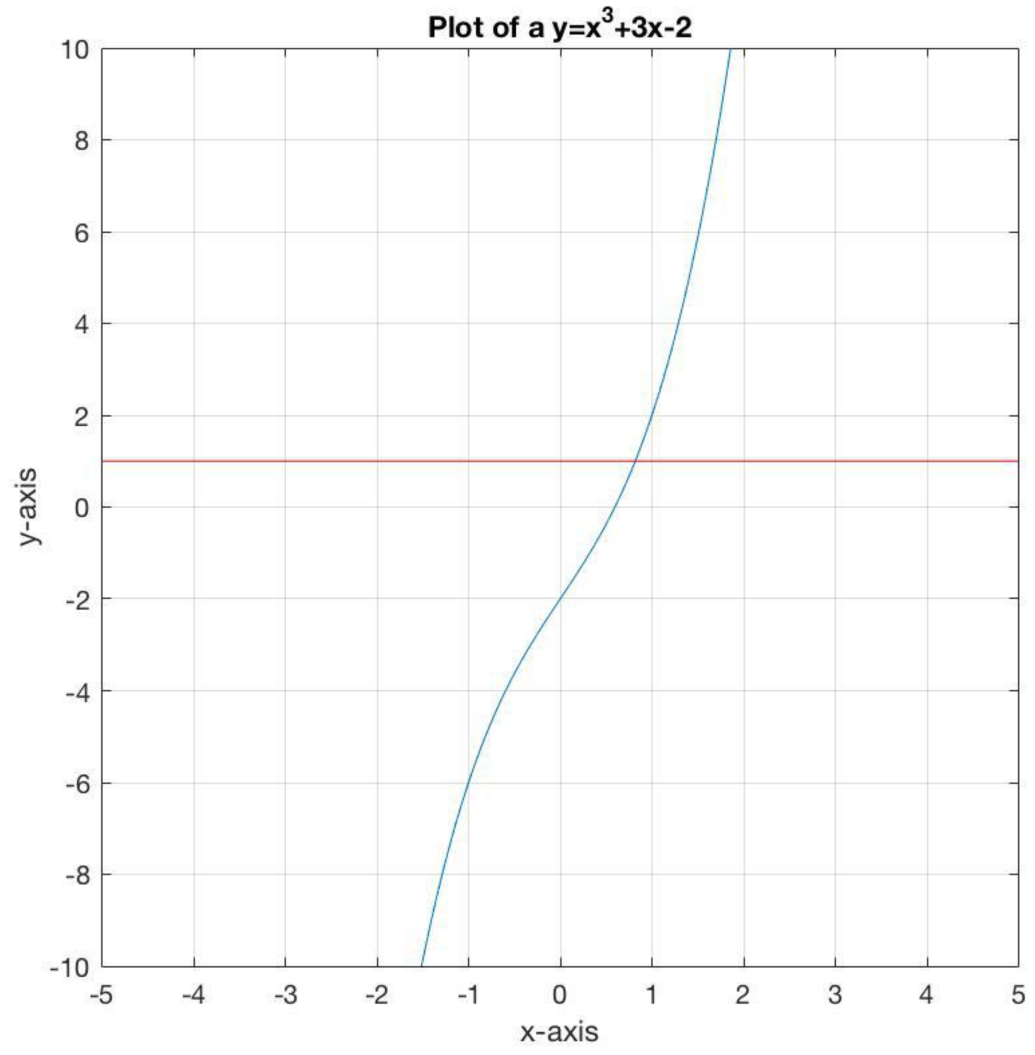

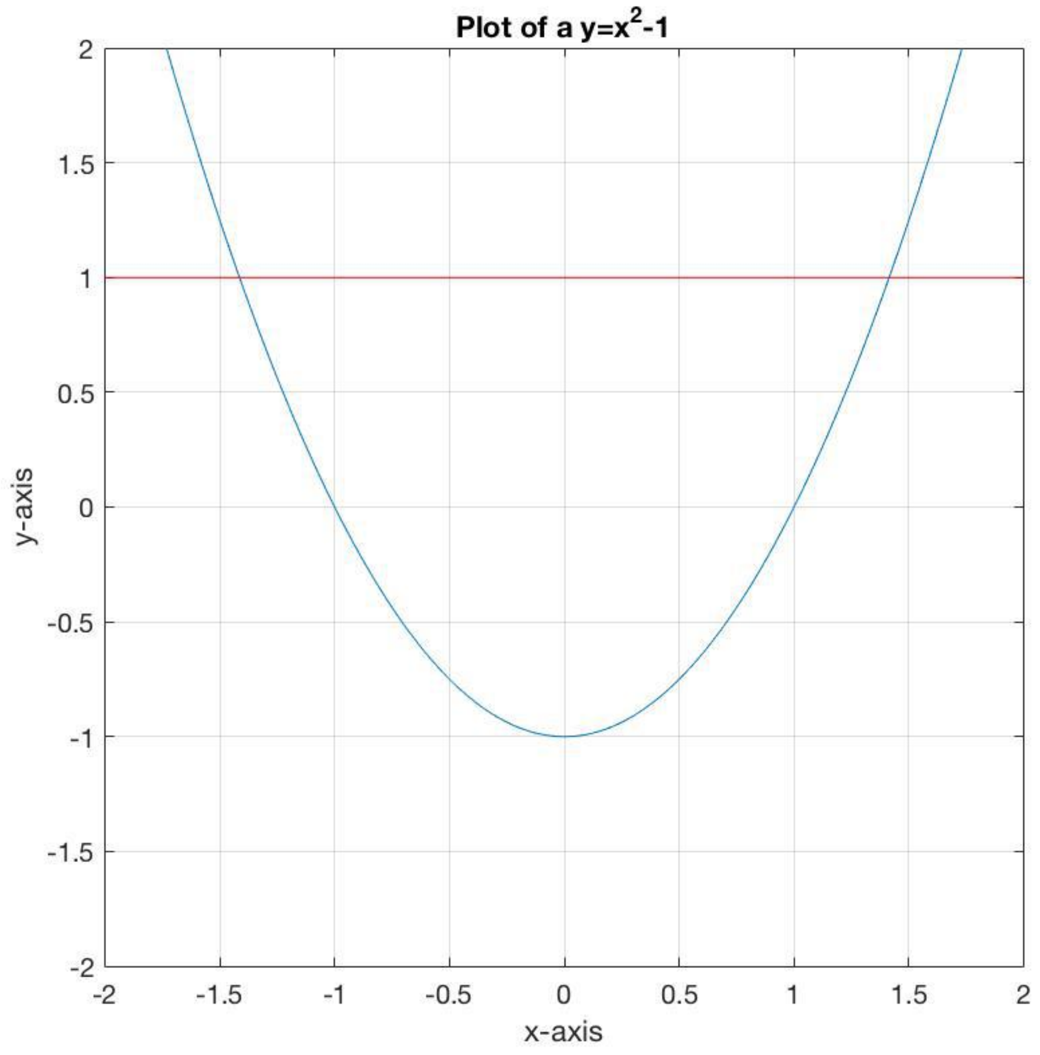

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1.5.1 Linear Growth

Linear and Exponential Growth

- **One can use a variety of functions to *model* real world phenomenon in science, business, economics, public policy, and more.**
- **Models can be very complicated, but two relatively simple starting points are *linear* and *exponential* models.**
- **In linear models, rate of change is constant.**
- **In exponential models, rate of doubling is constant.**

Linear Equations

- Equations of the form $y = ax + b$; a is the slope or rate of growth.
- We want to compute values of x given y and vice versa.
- Sometimes we need to perform some algebraic rearrangements first

Suppose it costs \$20 to manufacture a widget, with a start-up cost of \$1,000. Write the cost function in terms of number of widgets manufactured.

1.5.2 Exponential Growth

Exponential Models

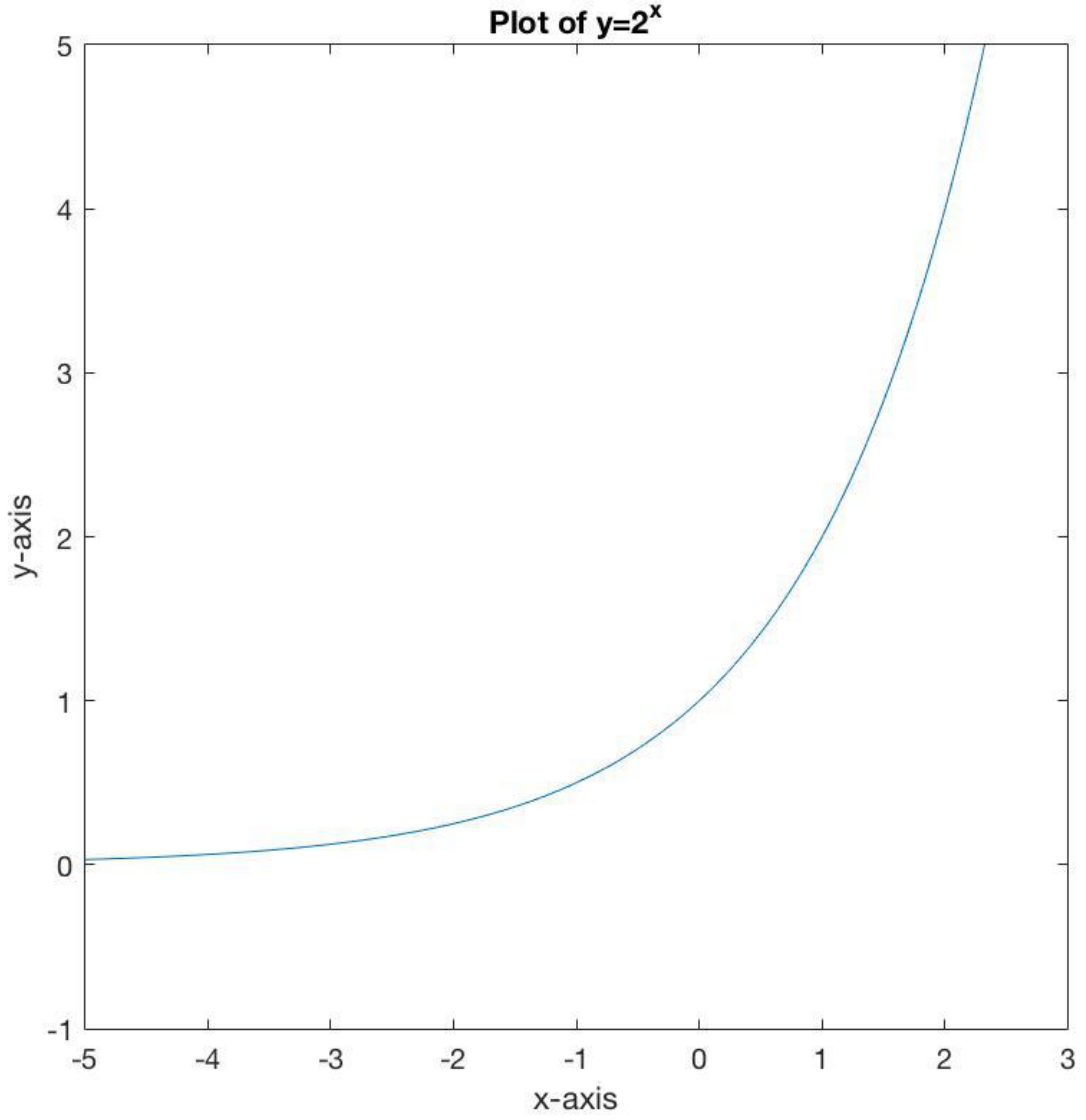
$$f(x) = a^x$$

- These may look daunting! However, we can use our exponential and logarithmic properties (tricks) to make our lives easier.
- Recall that $$y = a^x \Leftrightarrow \log_a(y) = x$$
- From this, we can approach many equations that look intimidating.

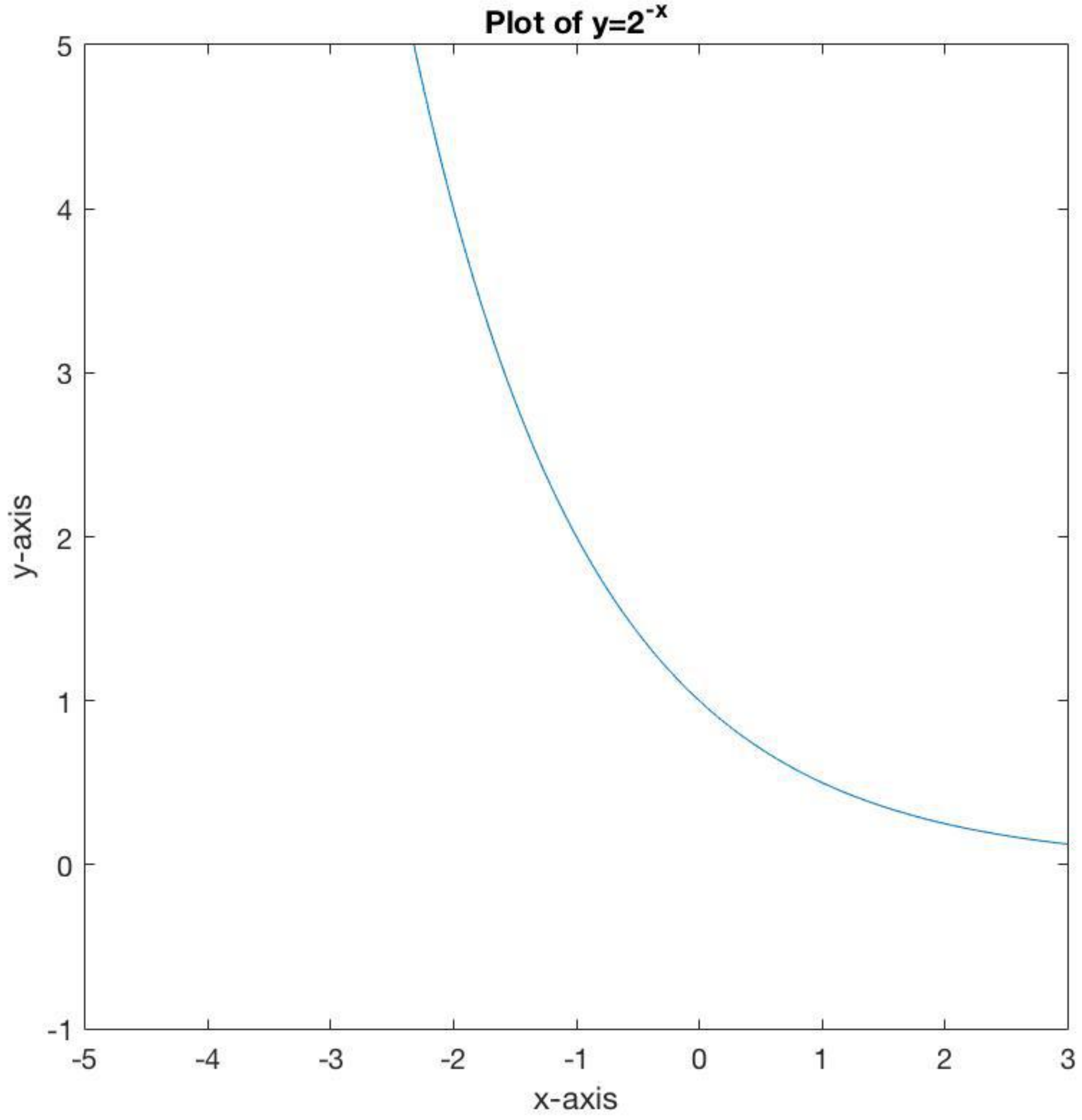
Properties of Exponents

Basic Rules:

- $a^{x+y} = a^x a^y$ **(same base, different exponents)**
- $a^x b^x = (ab)^x$ **(different base, same exponent)**
- $(a^x)^y = a^{xy}$ **(iterated exponents)**
- $x^0 = 1$ **for any value of x** **(convention)**



Plot of y=2^x
5
4
3
2
1
0
-1
-5
-4
-3
-2
-1
0
1
2
3
y-axis
x-axis



Plot of y=2^-x
5
4
3
2
1
0
-1
y-axis
-5
-4
-3
-2
-1
0
1
2
3
x-axis

Suppose a population of javelinas doubles in size every 16 months. If there are 100 to start,

A. Write the population function in terms of time in months.
B. What is the population of javelinas after 6 years?
C. How long until there are 1000 javelinas?

2. Counting and Probability

2.1.1 Factorials

2.1.2 Combinatorics

2.2.1 Probability Theory

2.2.2 Probability Examples

2.1.1 Factorials

Combinatorics

- Combinatorics is the mathematics of counting. It can be quite delicate.

- Two important notions for us are those of combinations and permutations.

- In order to define these, we need the notion of factorial and binomial coefficient.

Factorial!

- The factorial of a number is simply the product of itself with all positive integers less than it.
- By convention, $0! = 1$.
- It is possible to define the factorial for non-integers, but this quite advanced and is not part of the CLEP.
- When computing with factorials, it is helpful to *write out the multiplication explicitly*, as there are often cancellations to be made.

Formulas and Notations

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$$1! = 1$$

$$2! = 2$$

$$3! = 6$$

$$4! = 24$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Compute the following:

$$5!$$

$$\frac{6!}{5!}$$

$$\frac{n!}{(n-1)!}$$

$$\frac{0!}{2!}$$

$$\binom{6}{2}$$

$$\binom{3}{3}$$

$$\binom{7}{0}$$

$$\binom{5}{3}$$

2.1.2 Combinatorics

Counting with Factorials

- Factorials are useful for *combinatorics*, i.e. problems involving counting.
- Given n objects, the number of groups of size k when *order doesn't matter* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Given n objects, the number of groups of size k when *order matters* is $\frac{n!}{(n-k)}$.

How many groups of 3 from 10 are possible, if order matters?

If order doesn’t matter?

How many codes of length 2 may be generated from the letters {A,B,C,D,E,D} if the letters cannot be repeated, and order matters?

What if repeats were allowed?

2.2.1 Probability Theory

Probability

- Probability in mathematics quantifies random events.
- It is a large subject with a rich history; we focus on a few basic ideas.
- We consider the probability that *events* occur:

$$\mathbb{P}(A)$$

- Our goal is to understand how to compute such probabilities.
- Note that all probabilities have values between 0 and 1.

Notation

- Let A, B be any two events.
- The *union* of A, B is the event that either $A \text{ or } B$ occurs. It is denoted $A \cup B$.
- The *intersection* of A, B is the event the that both $A \text{ and } B$ occur. It is denoted $A \cap B$.
- The *complement* of A is the event that A does not occur. It is denoted A^c.
- There are some principle that dictate how to compute these quantities.

Basic Principles

- **Union Law:**

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- **Events A, B are *independent* if:**

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

- **The *conditional probability of* A *on* B *is***

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- **The *law of the complement* states**

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

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2.2.2 Probability Examples

Draw one card from a full deck of 52, well-shuffled.

$$\mathbb{P}(\text{diamond or Queen})$$

$$\mathbb{P}(\text{ace or King})$$

$$\mathbb{P}(\text{diamond or club})$$

Roll a die twice

$$\mathbb{P}(\text{roll } 1 \leq 2 \text{ and roll } 2 \leq 3)$$

$$\mathbb{P}(\text{sum of rolls } = 7)$$

$\mathbb{P}(\text{second roll different from first})$

For two events A, B, suppose $\mathbb{P}(A) = \frac{1}{2}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(A \cap B) = \frac{1}{3}$.

Are A, B independent?

3. Data Analysis and Statistics

3.1 Visual Analysis of Data

3.2.1 Basic Statistics Examples

3.2.2 Basic Statistical Theory

3.3 Normal Distributions

3.4 Bivariate Data

3.1 Visual Analysis of Data

Visual Analysis of Data

- **When analyzing data, finding a visual display is a useful first step.**

- **Various graphical and plotting methods may be of use. For the CLEP College Mathematics exam, one should be familiar with line graphs, bar graphs, histograms, pie charts, and stem plots.**

Bar Graphs and Histograms

- **Bar graphs and histograms use heights of rectangles to visualize quantities.**

- **Bar graphs are typically used for data that counts the number of things in a given *category*, while histograms are used for data that counts the number of things in a given *numerical range*.**

- **From our practical standpoint, there is little difference. Bar plots typically have space between the bars, while histograms do not.**

Pie Chart

- **Represent data categories as slices of a pie, with pie slice size proportional to proportion of data in a given category.**

A survey of 200 people are asked for their favorite color. The results are: blue (33), green (50), red (17), yellow (100). Display this data as a histogram and pie chart.

Line Plot

- Plot numerical data points, then connect points with line segments. Useful for pattern visualization.

Stem and Leaf Plots

- A rather odd method of displaying numerical data.
- Observations are sorted and displayed according to, for example, the tens digit and then the ones digit, though other digits could be used.

Make a stem and leaf plot for the following data:

$$\{6, 12, 24, 2, 3, 15, 11, 6, 8, 12, 23\}$$

3.2.1 Basic Statistical Theory

- **Statistics are used to summarize, analyze, or interpret data.**

- **All scientific and quantitative fields make use of statistics; we focus on a few basic examples.**

- **We consider in this section *measures of center* and *measures of spread.***

Measures of Center

Let $x_1, x_2, \ldots, x_n$ be a set of numerical data.

- **The *mean/average* of the data is given by**

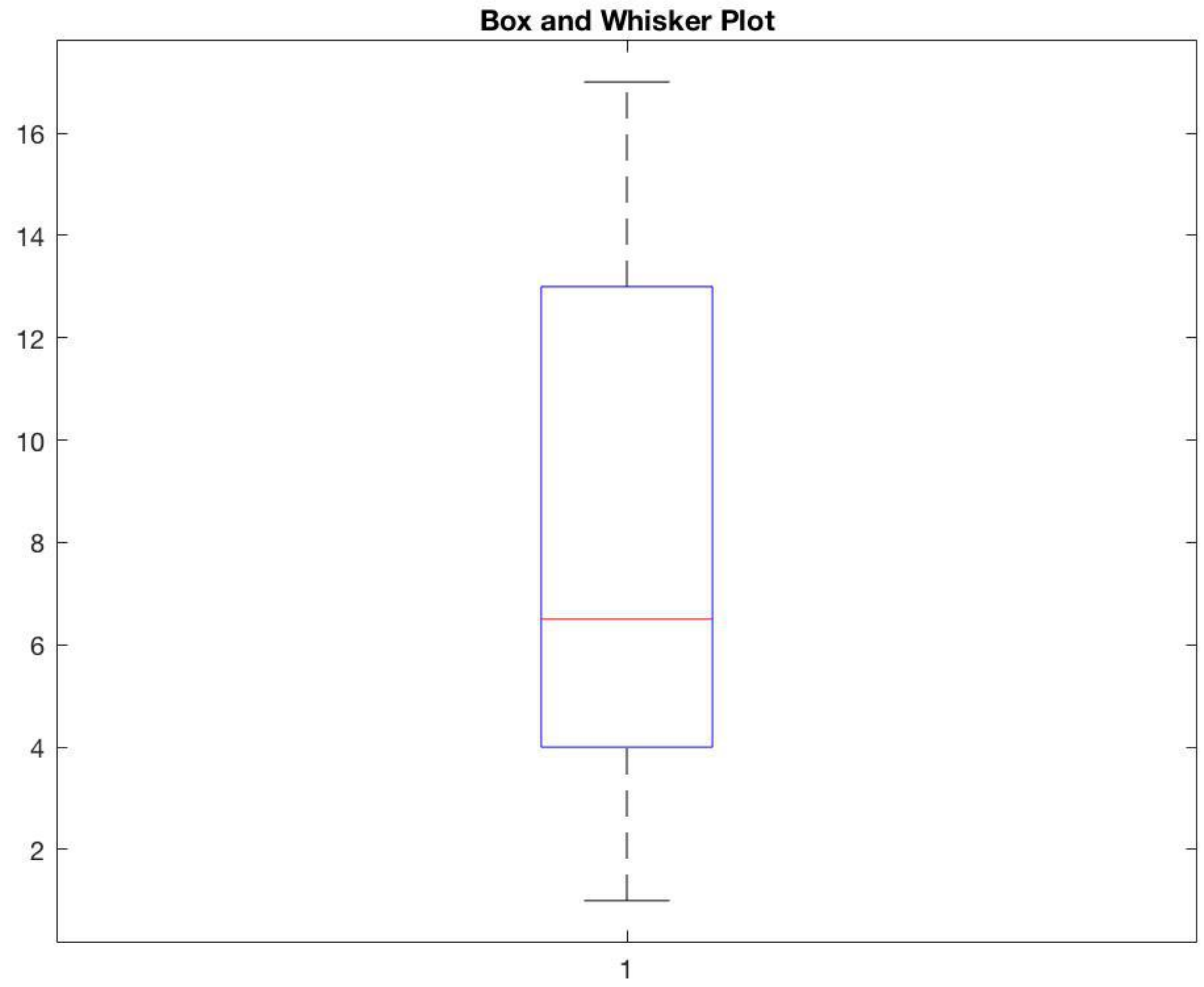
$$\frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k$$

- **The *median* of the data is given by first ordering the data, then taking the middle entry if n is odd or the average of the two middle entries if n is even.**
- **The *mode* of the data is given by the most commonly occurring data value.**

Measures of Spread

Let $x_1, x_2, ..., x_n$ be a set of numerical data.

- The *range* of the data is the largest value - smallest value.
- The *inner quartile range* is the range of the 50% of data closest to the median. That is, it is the 25^{th} percentile - the 75^{th} percentile.
- Measures of spread such as the above can often be visualized through plots, such as a *box and whisker plot*.


Box and Whisker Plot
16
14
12
10
8
6
4
2
1

Let $x_1, x_2, \ldots, x_n$ **be a set of numerical data.**

- **The *variance* of the data is given by**

$$\frac{1}{n}\sum_{k=1}^{n}(x_k-\bar{x})^2, \text{ where } \bar{x} \text{ is the mean}$$

- **The *standard deviation* of the data is the square root of the variance**

$$\sqrt{\frac{1}{n}\sum_{k=1}^{n}(x_k-\bar{x})^2}, \text{ where } \bar{x} \text{ is the mean}$$

3.2.2 Basic Statistics Examples

Compute the mean, median, mode, variance, standard deviation, 25th and 75th percentiles for the following data:

$$\{3, 4, 8, 10, 10, 12, 18, 30\}$$

$$\{6, -2, 4, 8, 0, 0, 3, 0, -7, 5, 9, -1\}$$

Compute all of the above statistics, and also give a box and whiskers/boxplot for the following data:

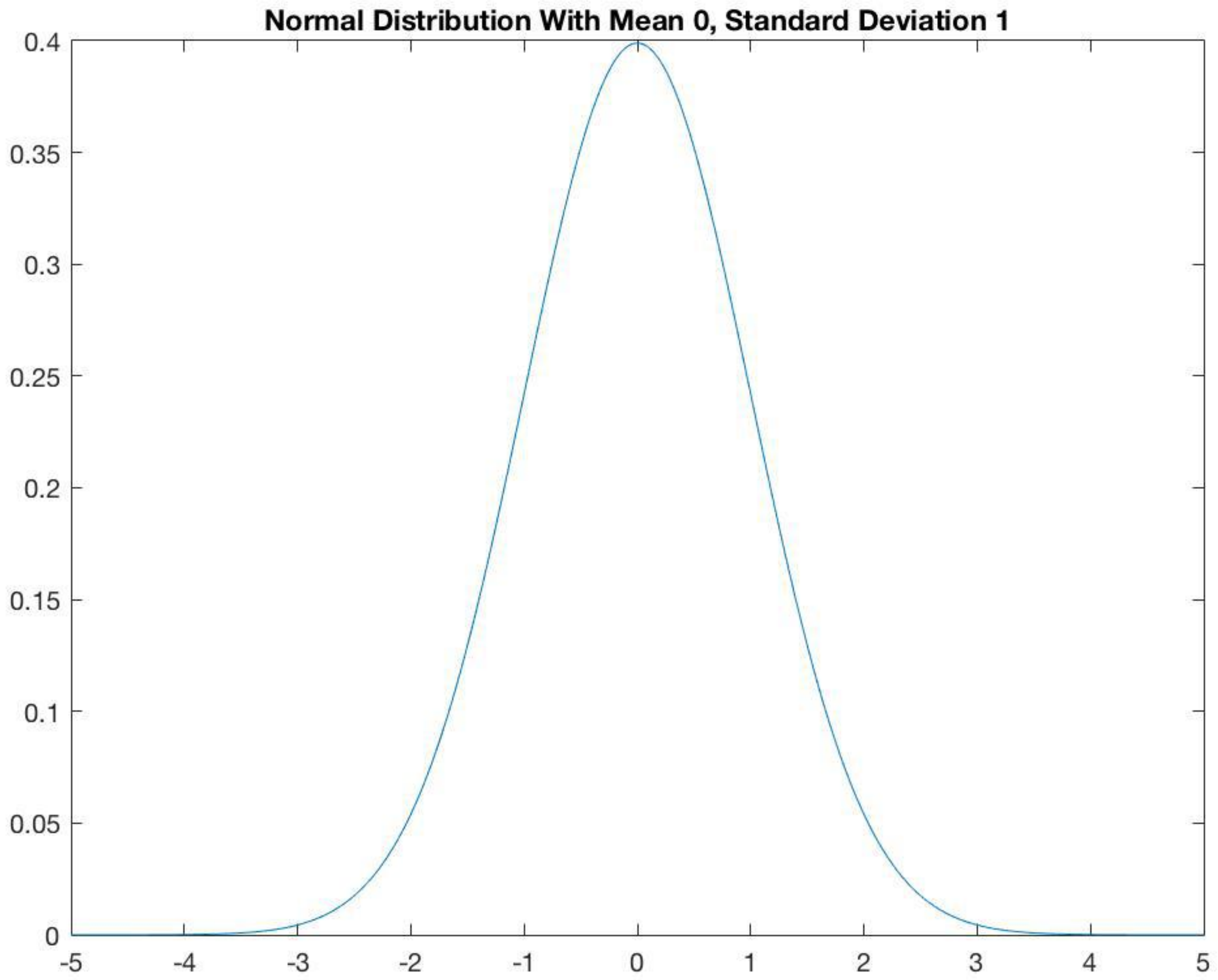
$$\{7, 9, 1, 0, -3, -5, 1, -1, 2, 2, -2, 6, 0, 5\}$$

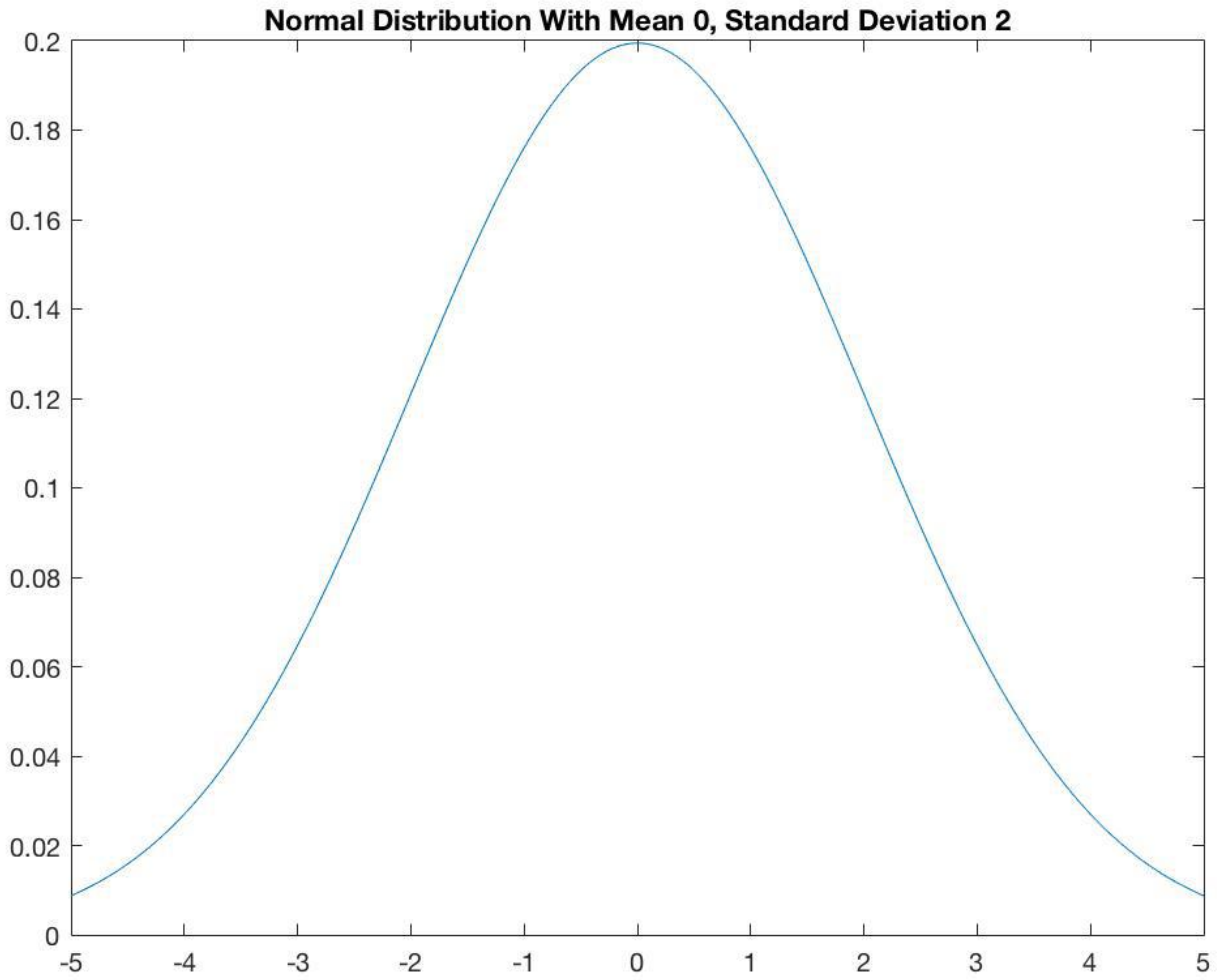
3.3 Normal Distributions

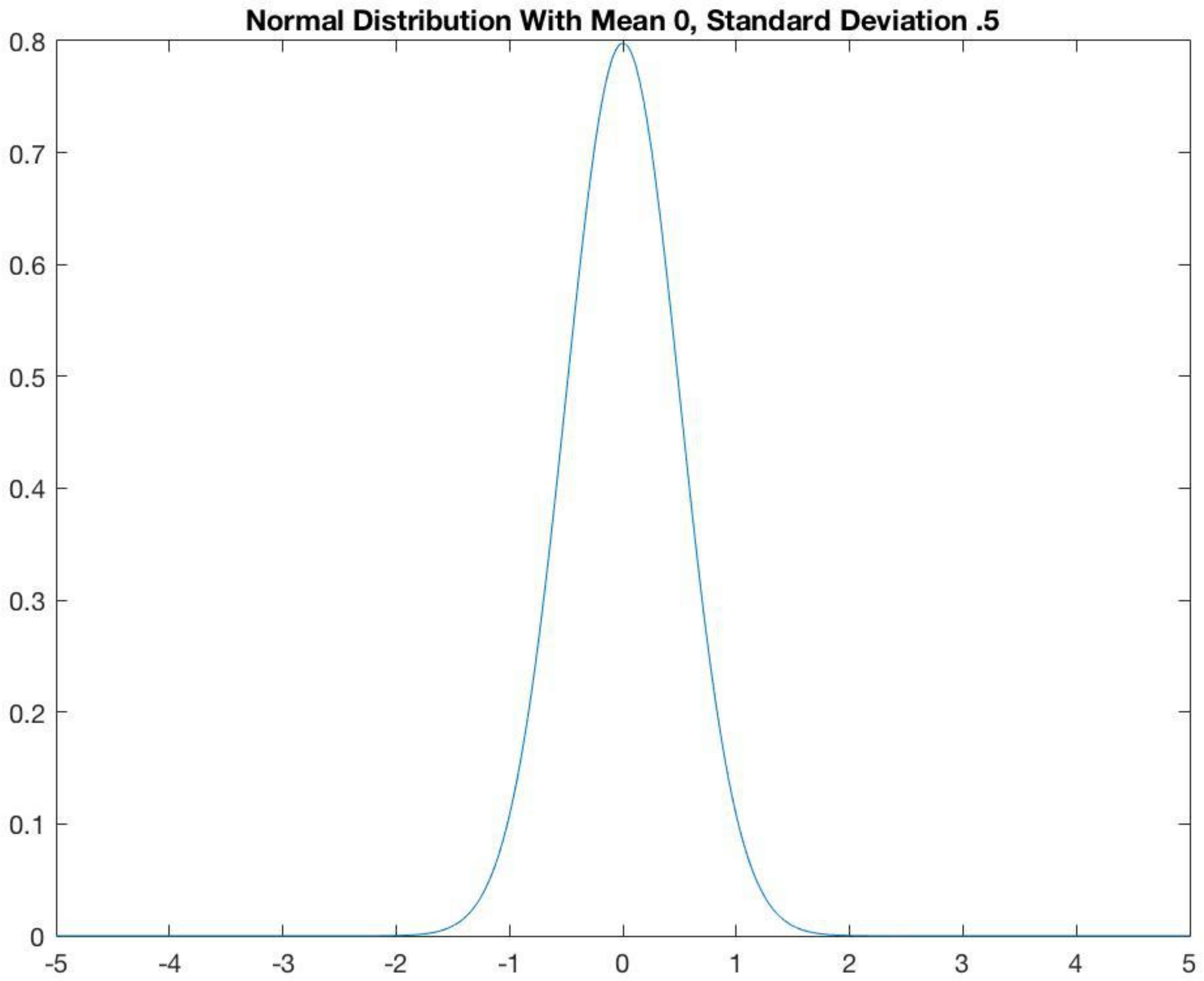
- **Normal distributions are classic examples of *continuous probability distributions.***

- **They are used as models in a variety of quantitative disciplines, and are easy to manipulate (compared to many other continuous probability distributions.)**

- **Geometrically, they look like bells, hence the name *bell curve.***


Normal Distribution With Mean 0, Standard Deviation 1
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


Normal Distribution With Mean 0, Standard Deviation 2
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


Normal Distribution With Mean 0, Standard Deviation .5
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

Sort the following normal distributions from greatest to smallest variance:

- **Normal distributions are centered at their mean.**

- **The standard deviation determines how thin (small standard deviation) or fat (large standard deviation) the shape.**

- **The total area under the curve = 1; this can be shown rigorously with integral calculus.**

- **Suppose a quantity** X **is random and follows a normal distribution. Then:**

$$\mathbb{P}(a \leq X \leq b) = \text{Area under curve between } x = a \text{ and } x = b$$

68-95-99.7 Rule

- **Precise computations for normal distributions require calculus. As a substitute, we often use the *68-95-99.7 rule.***
- **Suppose X is normal with mean μ and standard deviation σ.**
- $\mathbb{P}(\mu - \sigma \leq X \leq \mu + \sigma) \approx .68$

- $\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx .95$

$$\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx .997$$

Suppose height is normally distributed with mean 68” and standard deviation 3”. Estimate the following probabilities:

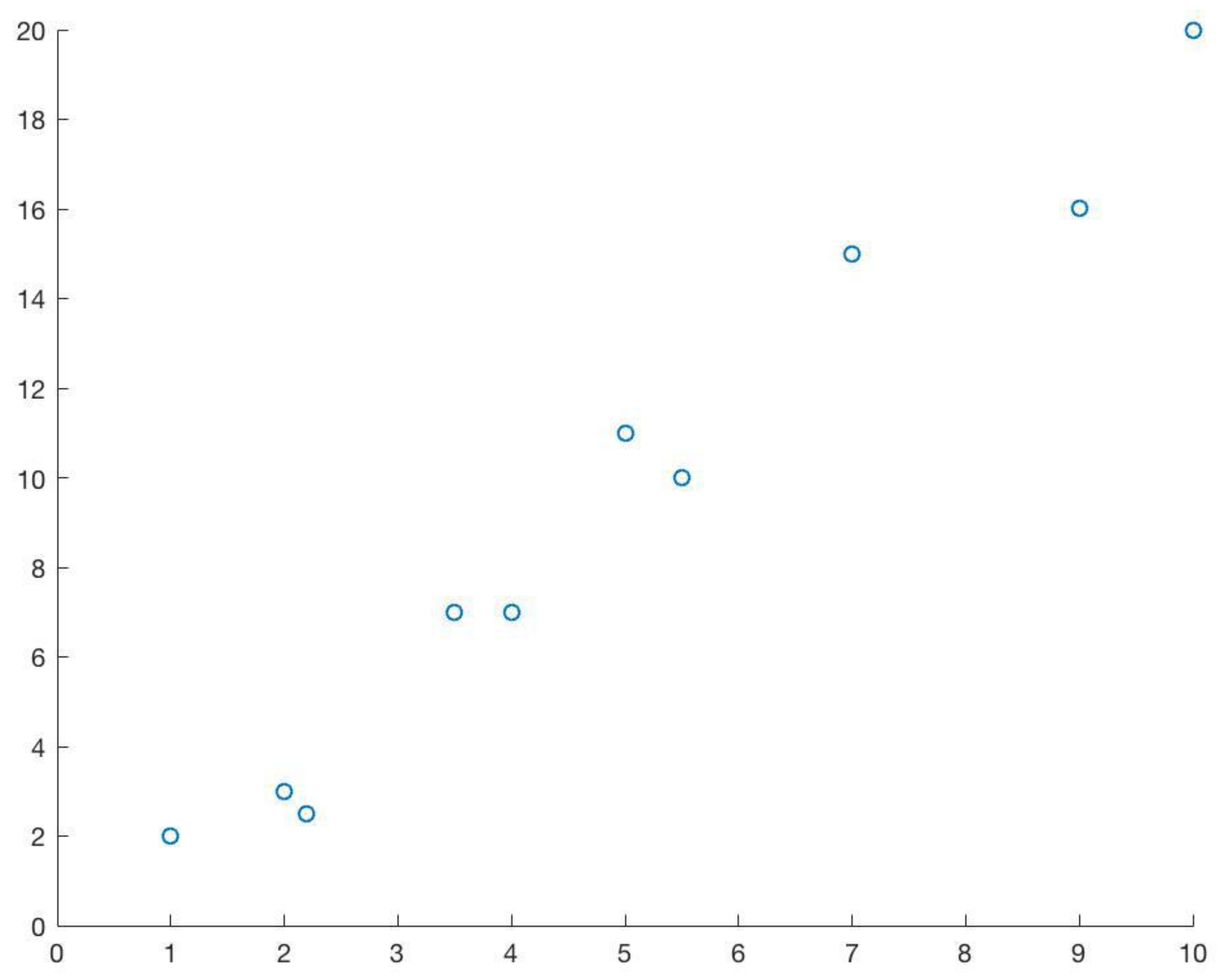
$$\mathbb{P}(\text{a person is between 65 and 71 inches})$$

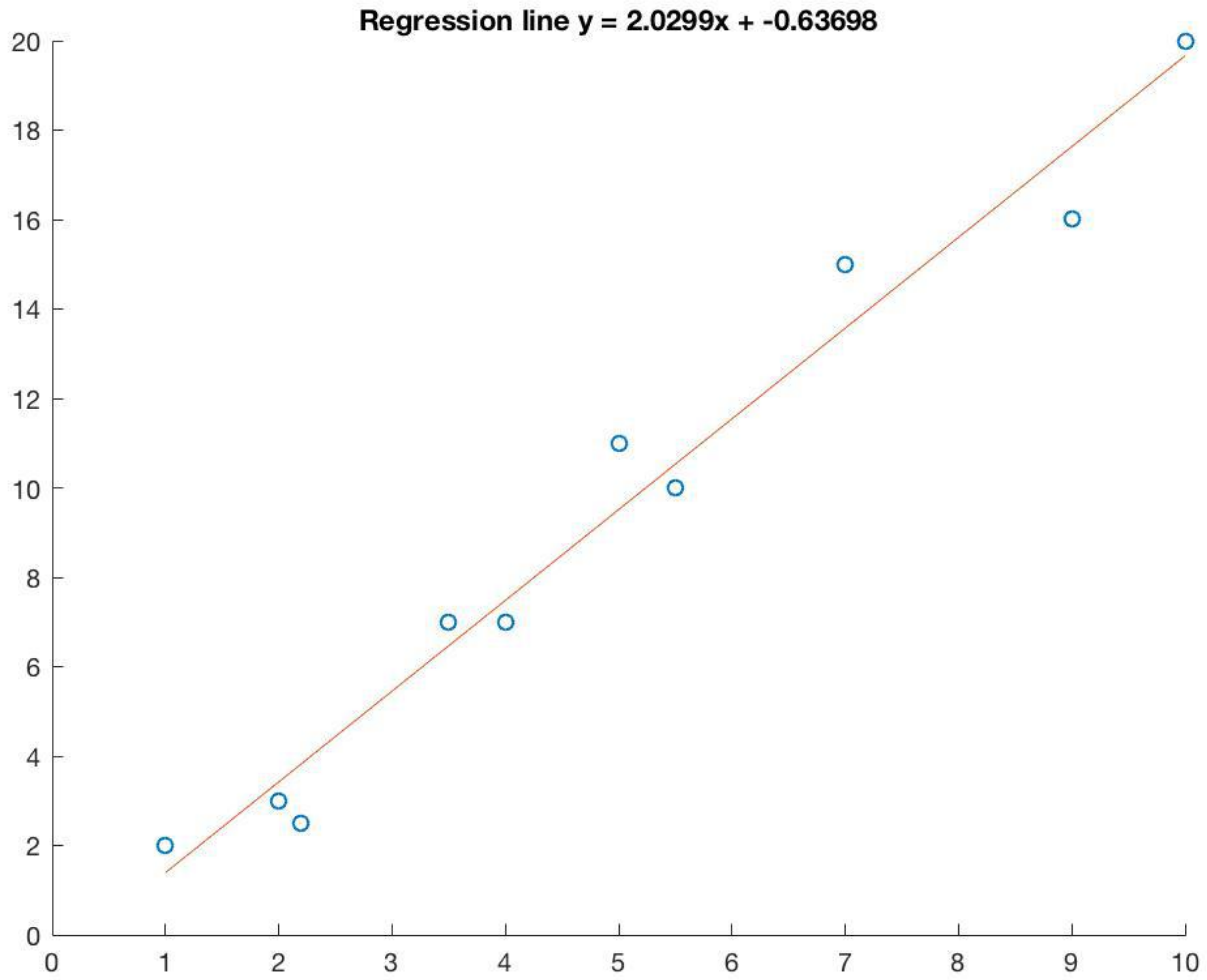
$$\mathbb{P}(\text{a person is } \geq 74 \text{ inches})$$

3.4 Bivariate Data

Bivariate Data

- **A classical problem in statistics is to fit a line to data points. Such a line is called a *regression line*.**




Regression line y = 2.0299x + -0.63698
0
2
4
6
8
10
12
14
16
18
20
0
1
2
3
4
5
6
7
8
9
10

Interpretation of Linear Regression

- Positive regression line slope indicates a *positive* correlation between the two variables.
- Negative regression slope indicates a *negative* correlation between the two variables.
- Not all variables have a linear relationship; in fact, it is quite rare.
- There are other regression methods, using more complicated functions, but these are beyond the scope of this course.
- The *correlation coefficient* can be used to quantify the strength of the linear relationship, but it is not covered on the CLEP.

For each of the following bivariate data samples, describe the trend and state if there is an evident linear pattern.

The weight of dogs is recorded as a function of time; the data and regression line appear below. The line has formula:

$$y = 8x + 7$$

Interpret the quality of line fit.

Interpret the slope and y-intercept of the line.

4. Financial Mathematics

4.1 Basic Financial Mathematics

4.2 Interest

4.3 Present and Future Value

4.1 Basic Financial Mathematics

Basic Financial Mathematics

- **In this section, we introduce terminology that may already be familiar to many students.**

- **Our goal is to mathematize familiar concepts.**

- **A *rate* is a quantity per unit measurement. It is usually calculated by a division.**

Taxes

- **A tax is an amount collected on income or a transaction. It can be derived on the sale of a good or service, an individual or corporation's income, the income from holding a financial asset, or from the value of property held.**

- **It is usually calculated as a percentage of the money under consideration. The ratio of tax paid to principal about is the *effective rate:***

$$\text{Effective Rate} = \frac{\text{Tax Paid}}{\text{Principal Amount}}$$

For each of the following principals and taxed amounts, compute the effective tax rate:

$550, 73$

$10071, 2467$

$174000, 18611$

Mark-Up

- ***Mark-up*** **is the amount added to the cost of a good or service corresponding to the per-unit profit of the seller.**

- **It is computed simply as the sale cost minus the cost for the seller to acquire or manufacture the product:**

$$\text{Mark-Up} = \text{Selling Price} - \text{Cost}$$

- **Similarly, the *Mark-down* is the amount taken off the price by a seller, to encourage consumers to buy:**

$$\text{Mark-Down} = \text{Old Selling Price} - \text{New Selling Price}$$

Compete the mark-up/down and relative mark-up/down for each of the following price pairs, where the first number is the original price, and the second price is the modified price.

$$(600, 500)$$

$$(1500, 1750)$$

$$(25, 15)$$

$$(18500, 25000)$$

4.2 Interest

Interest

- **Interest is money paid by a borrower to a lender, beyond the initial amount lent.**

- **It may be understood in some cases as *the cost of borrowing money.***

- **Banks often pay interest to those who keep money with the bank.**

- **Loans (mortages, cars, short-term) typically have interest attached to them, representing the cost of borrowing the money.**

- **We consider a few fundamental interest models.**

Simple Interest

- ***Simple interest*** **computes the interest on the initial monetary amount (*the principal)* simple by multiplying by a fixed rate:**

$$I = P \times r \times t$$

- P **is the principal, i.e. the initially amount of money.**
- r **is the rate of interest. It depends on the type of loan, reliability of borrower and lender, and economic conditions.**
- t **is time.**

For each of the following, compute the simple interest earned on the principal:

$10000, 3.1\%, 5 \text{ years}$

$5500, 7.3\%, 6 \text{ months}$

$1000, 18\%, 10$ years

$100000, .1\%, 10$ years

Compound Interest

- **A slightly more delicate model for interest is that of *compound interest.***
- **In this model, interest is paid not just on the principal, but on the *interest accrued over time.***
- **The formula is a bit more complicated:**

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- **The new variable n is the number of times the interest earned is accounted into the principal, i.e. the number of times the interest is *compounded.***

Continuous Compounding

- **Instead of compounding a finite number of times, one can allow $n \to \infty$, corresponding to *compounding continuous.***

- **Under this model, the amount earned on the principal is $A = Pe^{rt}$.**

- **Here, $e \approx 2.71$ is *Euler's constant*, an infinitely long, non-repeating (irrational) number.**

For each of the following, compute the compound interest earned plus principal.

$1000, 4.5\%, 3 \text{ years, compounded annually}$

$3000, 7.5\%, 1 \text{ year, compounded quarterly}$

$3000, 7.5\%, 1$ year, compounded daily

$3000, 7.5\%, 1$ year, compounded continuously

For a principal of 1000 and a rate of 5%, compute the value after 10 years under the model of: simple interest, monthly compounding, and continuous compounding.

4.3 Present and Future Value

Present and Future Value

- **If one has a principal investment amount P that earns interest, it will be worth more than P at a future time.**

- **The amount it is worth depends on how the interest is computed, the interest rate, the time of investment, and so on.**

- **One can compute the *future value of the investment* by computing how much money would be generated from the principal according to a particular interest scheme. One can compare the future value to the value of the principal now, called the *present value.***

Suppose we want to have $20000 in 5 years. Assuming our investments will earn 5% interest, compounded yearly, how much should we invest today to meet our goal?

What about under continuous compounding?

Inflation

- **Future value indicates that money can become more valuable over time, if invested.**

- **The flip side is that if uninvested, money typically loses value over time due to *inflation*. This is the process by while prices tend to increase over time.**

- **Inflation is extremely well-studied in the field of economics, and has upsides as well as downsides.**

- **One should compare the future value of money to the future cost *after inflation*, in order to determine if the investment actually improves buying power or not.**

Suppose a house cost $10000 in 1950. Supposing inflation is 4% annually what would the cost be in today's dollars?

5. Geometry

5.1 Lines

5.2.1 General Triangles

5.2.2 Right Triangles

5.3.1 Quadrilaterials I

5.3.2 Quadrilaterals II

5.4 Circles

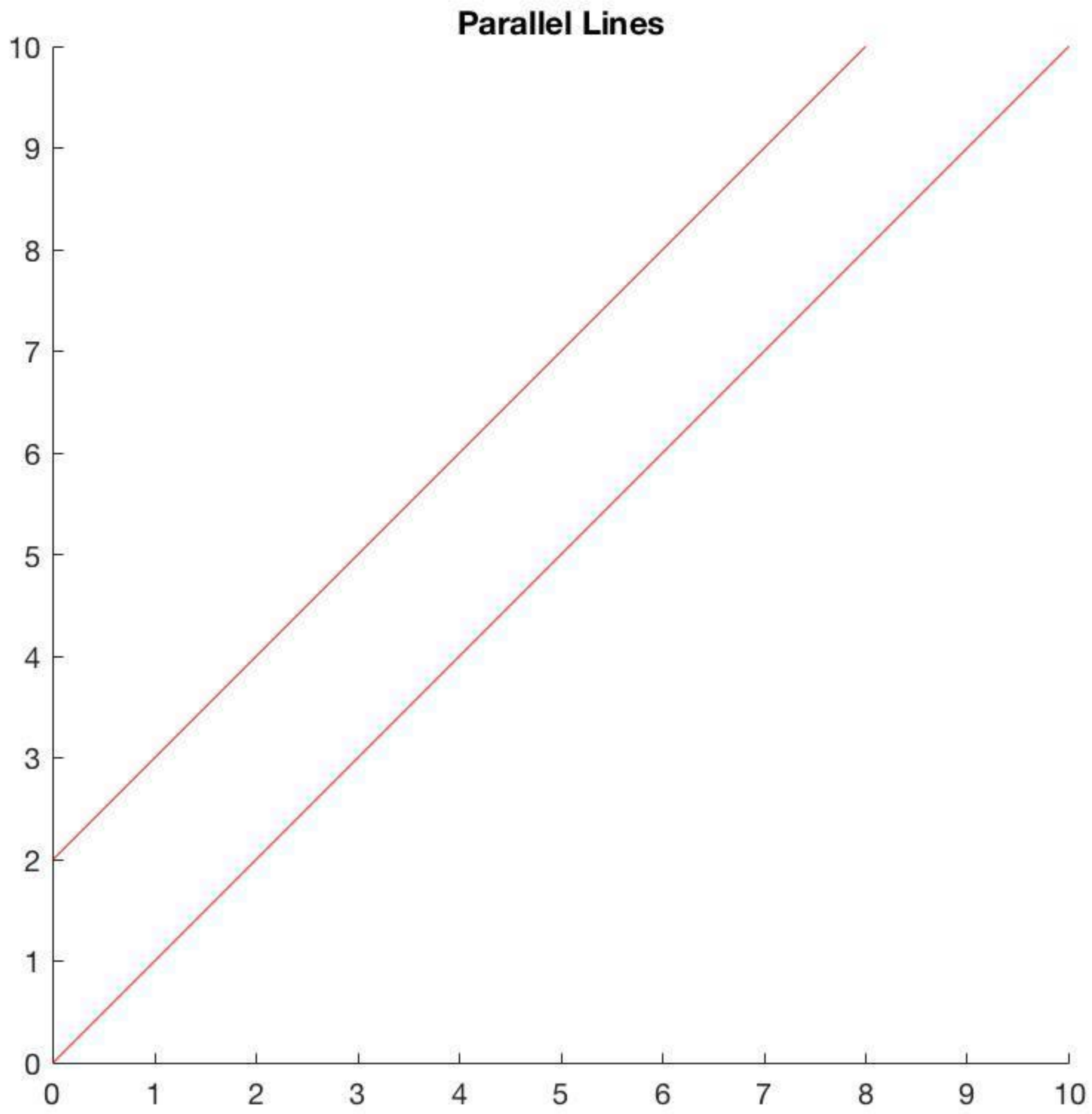
5.1 Lines

- The study of planar geometry goes back at least to Euclid (~300 BCE).

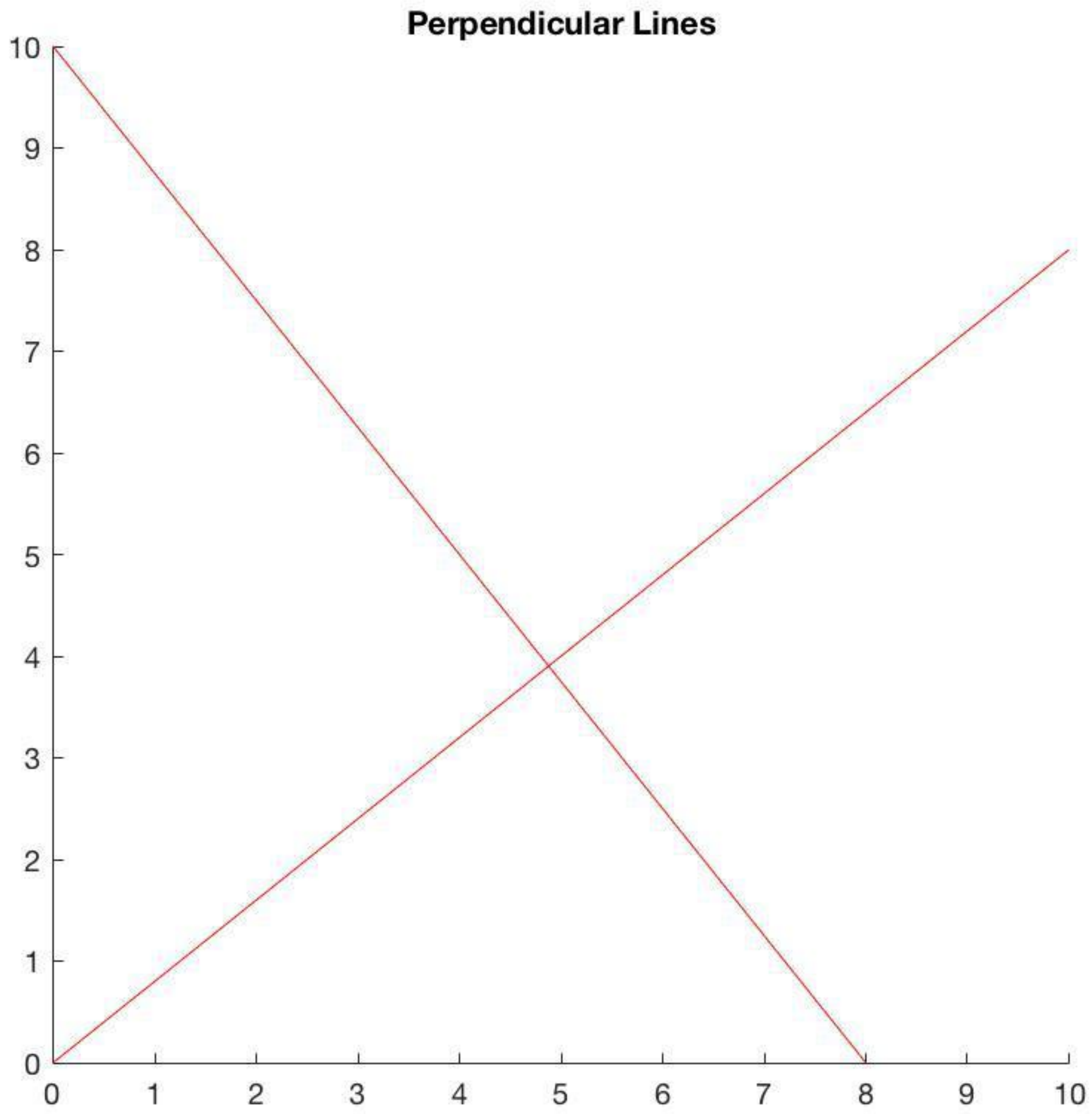
- Lines are an important part of this theory.

- Two lines are said to be *parallel* if they never intersect, or equivalently, if they have the same slope.

- Two lines are said to be *perpendicular* if they intersect at a right angle.



Parallel Lines
0
1
2
3
4
5
6
7
8
9
10


Perpendicular Lines
10
9
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8
9
10

Analysis of Slopes

- Given formulae for two lines, one can quickly determine if they are parallel or perpendicular by analyzing their slopes:

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

- The lines are *parallel* if $m_1 = m_2$.

- The lines are perpendicular if $m_1 = -\frac{1}{m_2}$.

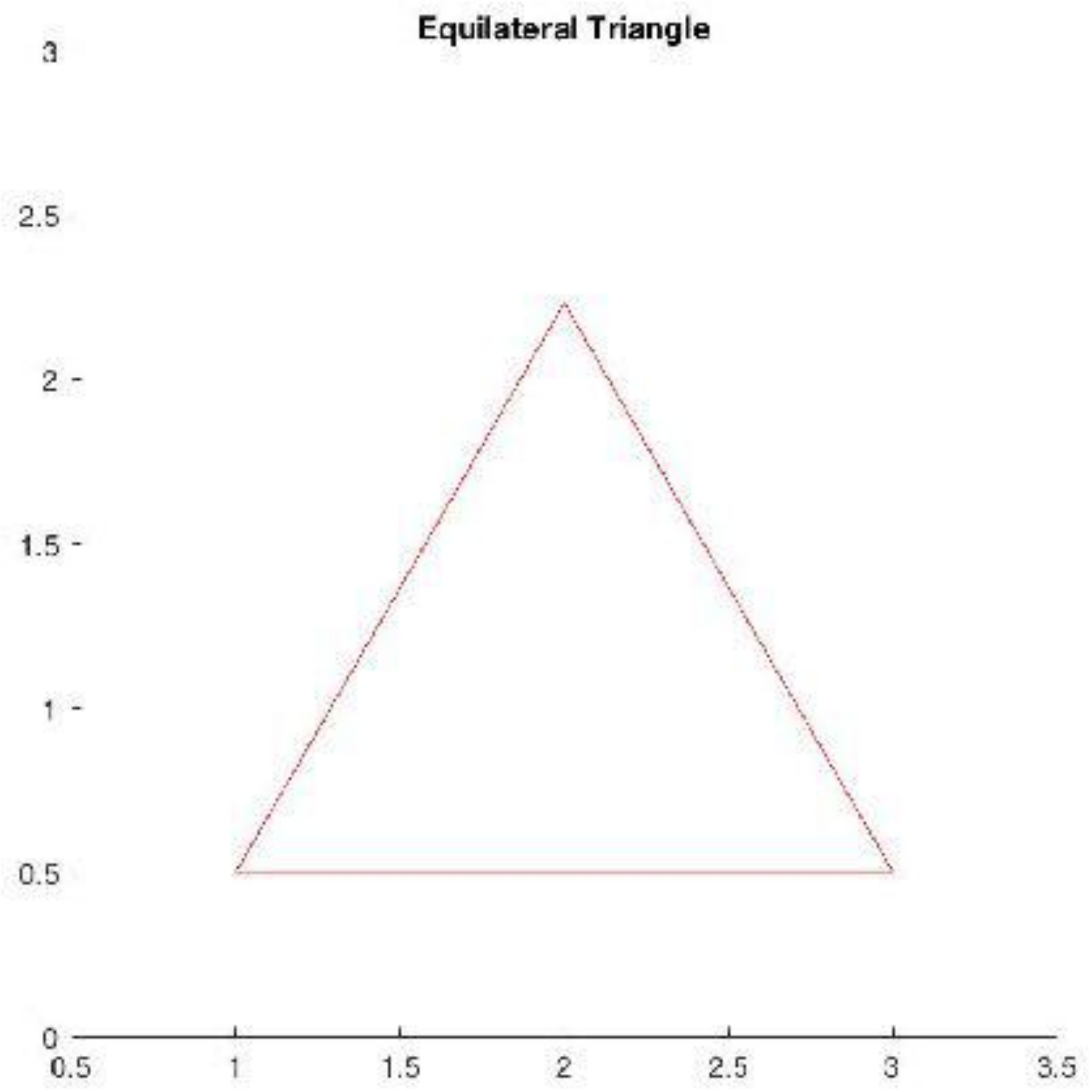
Determine if the following lines are parallel, perpendicular, or neither:

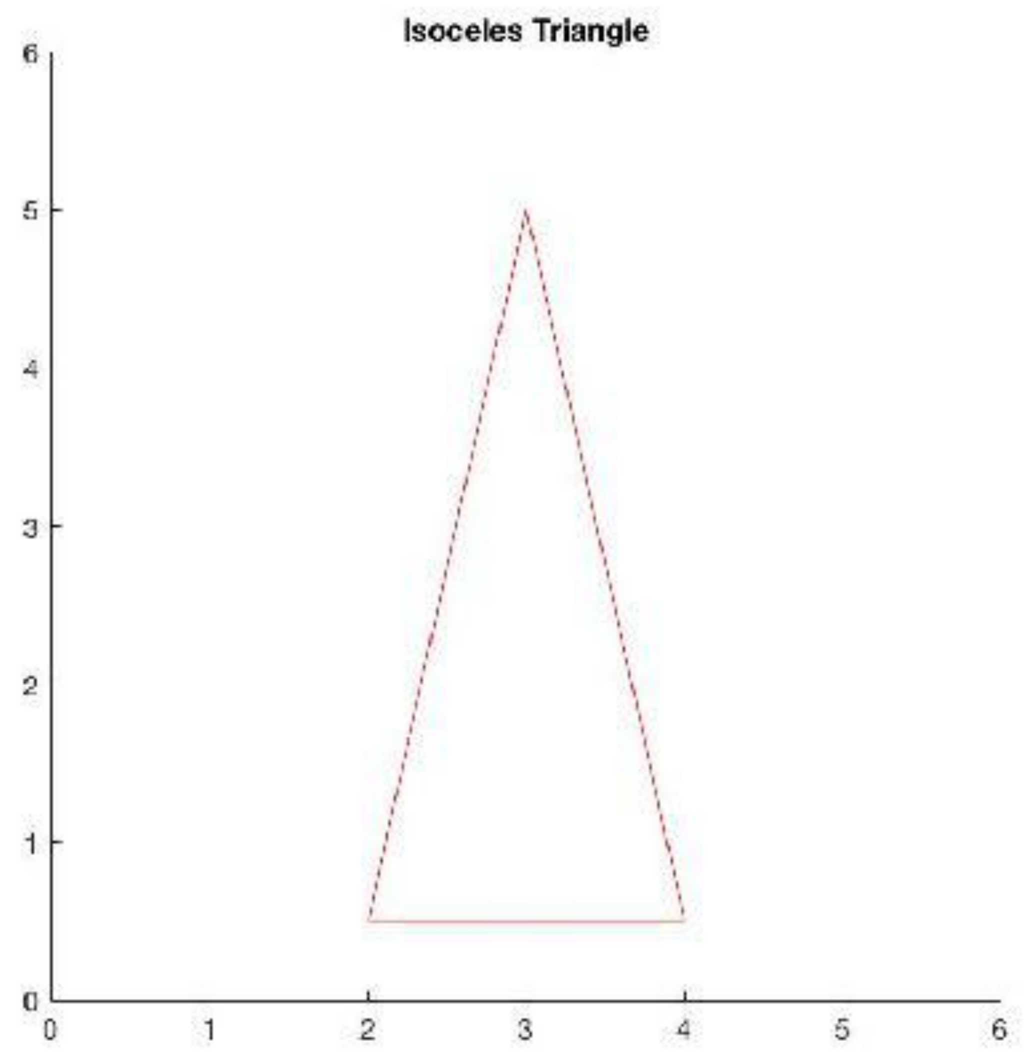
- $y=3x+1$, $y=3x-6$
- $y=3x+1$, $y=1/3x-2$
- $y=x+1$, $y=-x+2$
- $3y=4-x$, $2x=6-6y$

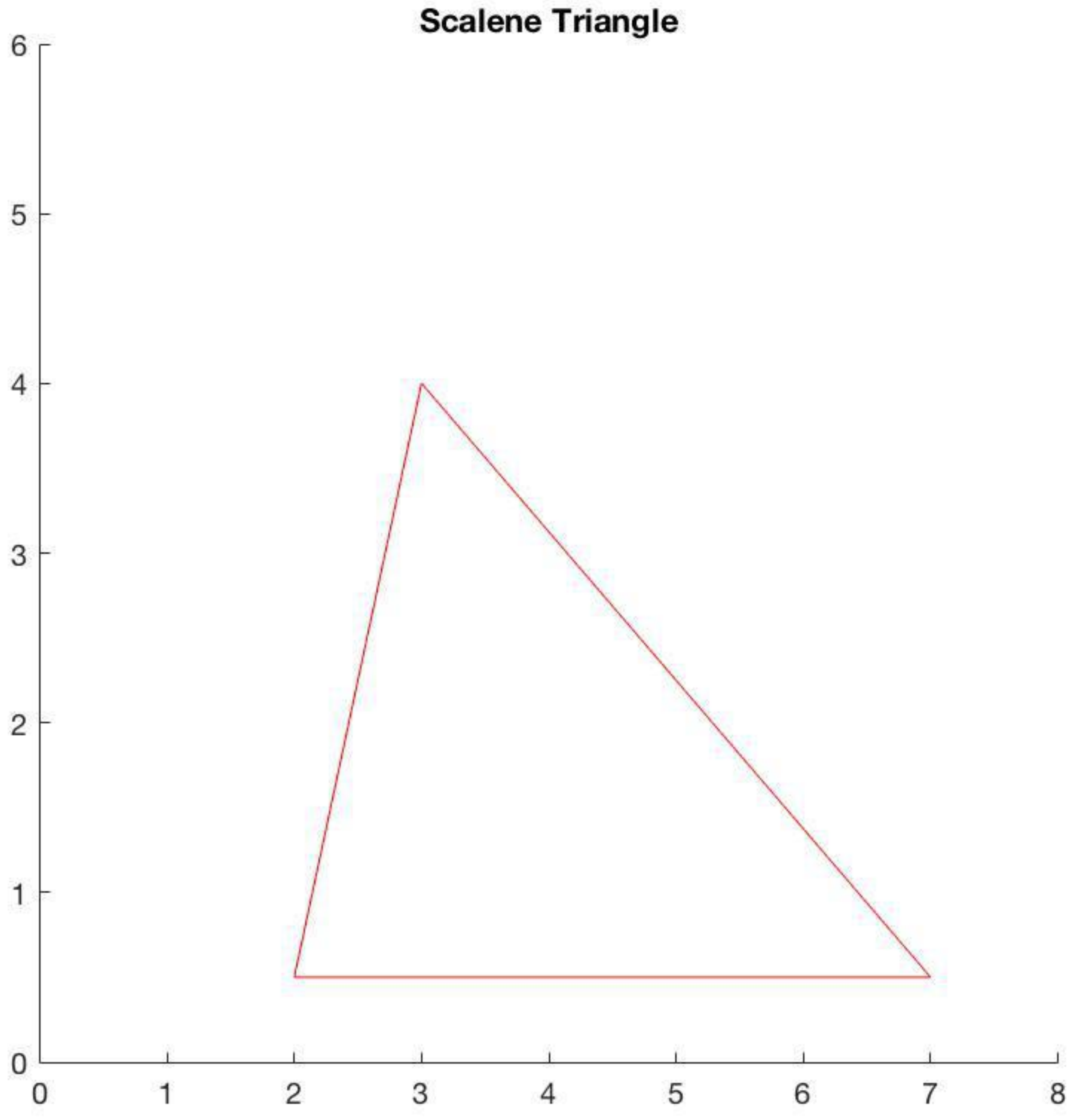
5.2.1 General Triangles

- **Triangles are among the simplest *polygons*.**
- **They are polygons with the fewest possible number of sides: 3.**
- **Triangles can be classified based on their angles or side lengths (it is equivalent to discuss one or the other!)**
- **Triangles also have several very nice formulas governing in them, particularly if one of the angles of the triangle is equal to 90°.**

Triangle classification by side length






Scalene Triangle
0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8

Area and Perimeter of Triangles

- **The *perimeter* of a polygon is the length of all its sides.**

- **The *area* of a polygon is the amount of space it encloses.**

- **For a triangle with *base length* B and *height length* H, the area of the triangle may be computed as:**

$$Area = \frac{1}{2}B \times H$$

Find the areas of the following triangles:

- **B=7, H=4**
- **B=8, H=4 (right triangle)**
- **B=2, H=2**

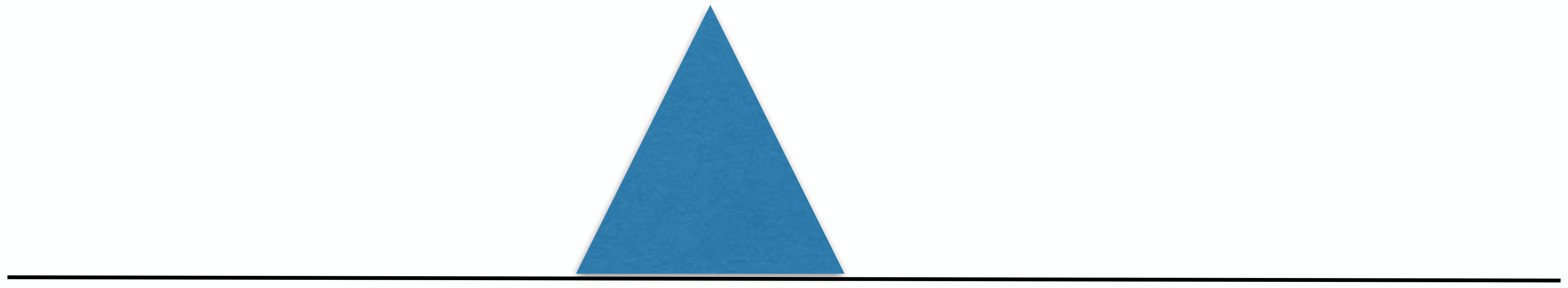
Regarding Angles

- The angles of a triangle must sum to 180°.
- An equilateral triangle has all angles of equal size. They are thus all of size $\frac{180}{3} = 60^\circ$.
- An *angular bisector* is a line that splits an angle into two equal parts.

Find the missing angles:

- 105, 40, ?

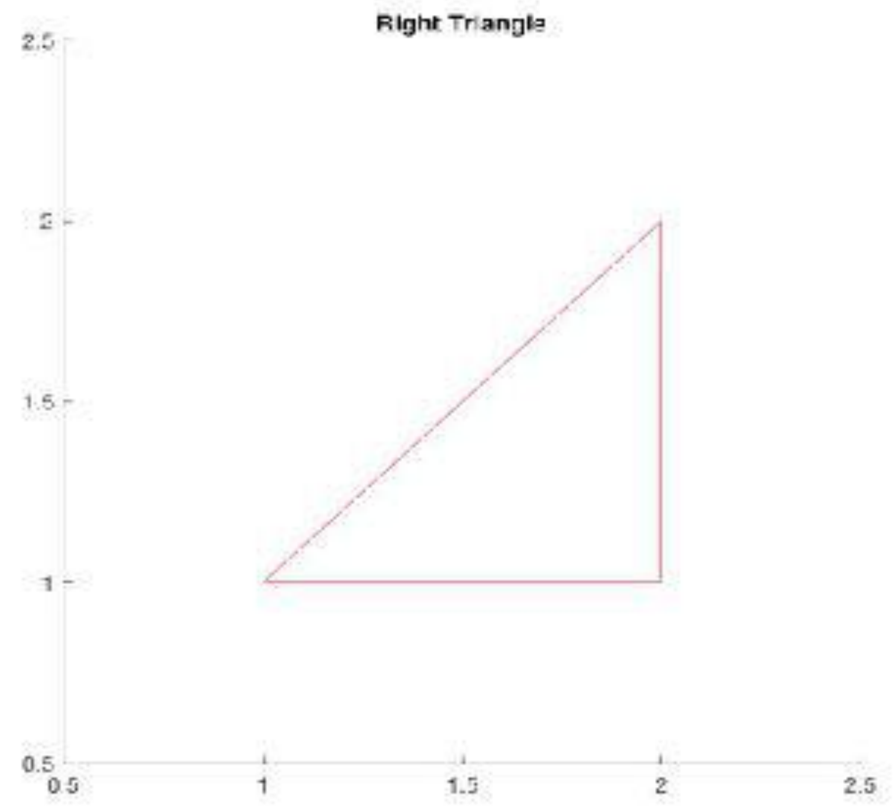
- 30, missing angles are opposite to equal side lengths (so use isosceles trick),
- Top angle=50, sides are same, want exterior angle.



5.2.2 Right Triangles

Pythagorean Theorem

- **A triangle with one angle of size 90° is called a *right triangle.***
- **Right triangles have a special rule governing the length of their sides, given as the famous Pythagorean theorem:**



$$a^2 + b^2 = c^2$$

Find the lengths of the missing sides:

- x,5,8
- x,4,5
- 5,7,x

Find the areas of the following triangles:

- B=3, Hyp=8, so H=sqrt(55)
- B=5, Hyp=7, so H=sqrt(24)

5.3.1 Quadrilaterals I

- **One can consider polygons besides triangles. Those with four sides are called *quadrilaterals.* Those with higher numbers of sides have specialized names, often with Greek roots.**

- **We will consider area formulae for special quadrilaterals: parallelograms, including rectangles, rhombi, squares, and a more general class of quadrilaterals called trapezoids.**

Parallelograms

- **Parallelograms are quadrilaterals with opposite sides parallel.**

- **Parallelograms include rectangles, rhombi, and squares as special cases.**

- **Parallelograms have area given by their base multiplied by their height or *altitude.***

$$Area = B \times H$$

- **Parallelograms have the property that adjacent angles sum to 180°, and opposite angles are equal.**

Find the area of the following parallelograms:

- B=4, H=4
- B=10, H=10

Find the missing angles:

- parallelogram, adjacent angle=30
- parallelogram, opposite angle=30
- parallelogram, our angle is x+50, adjacent angle is 2x-40

Rectangles

- Rectangles are parallelograms with all angles equal to 90°
- The length of diagonals of rectangles may be found using the Pythagorean Theorem:

$$D = \sqrt{B^2 + H^2}$$

Rhombi

- A *rhombus* is a parallelogram with all sides of equal length.

Squares

- A square is a quadrilateral that is both a rectangle and a rhombus.

- It is the simplest quadrilateral, and also the most constrained in its definition.

5.3.2 Quadrilaterals II

Trapezoids

- A trapezoid is a quadrilateral that has one pair of parallel sides.

- So, parallelograms are trapezoids, but not all trapezoids are parallelograms.

- For a trapezoid, the area is found by accounting for both of the two parallel *bases:*

$$Area = \left(\frac{B_1 + B_2}{2}\right) \times H$$

Find the areas of the following quadrilaterals:

- **Trap, B1=8, B2=6, H=7**
- **Trap, B1=10, B2=15, H=4**

Similar Polygons

- Two polygons are said to be similar if their angles are the same, in the same order.

- Equivalently, polygons are similar if, after a rotation, they are the same up to scaling.

- Similar polygons look like magnified of shrunk versions of each other.

- One can use this scaling ration to relate the areas, perimeters, diagonal lengths, side lengths, etc. of one polygon to the other.

5.4 Circles

- Circles are sets of points that are at fixed distance to a *center* point.

- The distance from points on the circle to this center is called the *radius* of the circle.

- Circles do not fit into the polygon regime, because circles do not have edges per se.

- They may be though of as having infinitely many edges in a certain sense, which can be made precise with calculus.

Area and Circumference of Circles

- **The area of a circle is given in terms of its radius:**

$$Area = \pi r^2$$

- **The length of the circle is typically called *circumference* rather than perimeter, and may be computed as**

$$C = 2\pi r$$

- **One may also easily discuss the circumference in terms of *diameter* of the circle. The diameter is the length of a line going across the circle and through the center.**

- **Hence, the diameter of a circle has length equal to twice that of the radius:**

$$D = 2r$$

- **With this, we see that the circumference may also be computed in terms of diameter as**

$$C = \pi D$$

Find the area and circumference of the following circles:

- $r = 3$
- $d = 10$

Arcs in a Circle

- One can discuss inscribed angles in a circle, and the corresponding arc length they cut off.

- The size of the angle is proportional to the length of the arc:

$$\frac{\text{Length Arc}}{\text{C}} = \frac{\text{Angle}}{360}.$$

- A similar principle holds for wedge areas:

$$\frac{\text{Area Wedge}}{\text{Area Circle}} = \frac{\text{Angle}}{360}.$$

Find the area of the following circular arcs:

- **r=5, theta=90**
- **arc length =10, theta = 30**

6. Logic and Sets

6.1 Logical Statements

6.2.1 Set Theory I

6.2.2 Set Theory II

6.1 Logical Statements

Logical Statements

- Mathematical logic is an odd subject. It has roots in both mathematics and the physical sciences, but also philosophy.
- It had a renaissance in the early to mid-twentieth century.
- We focus on just a few basic idea.

Formalities

- A *sentence* is a statement that is either true or false.

- A sentence is *consistent* if it is not self-contradictory.

- *Truth* in logic relates only to the formal aspects of a sentence, rather than to its content.

- It is possible for a statement to be neither logically true nor logically false, since all of its meaning comes from outside the world of logic.

Statements

- *Statements* are sentences that are either logically true or logically false.

- Statements may be combined and modified in several important ways.

- Let A, B represent two statements.

- $A \vee B$ denotes A or B
- $A \wedge B$ denotes A and B
- $\sim A$ denotes not A
- **Note that:** $\sim (A \vee B) = (\sim A) \wedge (\sim B)$
- **Note that:** $\sim (A \wedge B) = (\sim A) \vee (\sim B)$

Conditional Statements

- **Given statements** $A, B,$ the conditional $A \to B$ **means** if A then B.
- The converse of $A \to B$ is $B \to A$.
- The inverse of $A \to B$ is $\sim A \to \sim B$.
- The contrapositive of $A \to B$ is $\sim B \to \sim A$.
- **The statement $A \to B$ is logically equivalent to its contrapositive, but not necessarily to its converse or inverse.**

Let $A = \text{Today is Thanksgiving}$
$B = \text{Today is a Thursday}$

Interpret the statement $A \to B$.

State and interpret the converse, inverse, and contrapositive of this statement.

If-And-Only-If

- A special kind of conditional statement is the *if-and-only-if* or *biconditional* statement.
- It is a statement of the form $(A \to B) \wedge (B \to A)$ denoted $A \leftrightarrow B$
- It means that A, B are logically equivalent, or that B happens exactly when A happens and vice versa.

Let $A = \text{Today is in April}$

$B = \text{Today is a Tuesday}$

$C = \text{Today is a Wednesday}$

Interpret the following:

$$A \vee B$$

$$(A \wedge B) \vee C$$

$$A \vee (B \wedge C)$$

$$(A \vee B) \wedge C$$

$$A \rightarrow B$$

$$\sim C \rightarrow B$$

$$C \rightarrow \sim B$$

6.2.1 Set Theory I

Basic Set Theory

- Set theory is intimately tied to mathematical logic.

- Great progress/confusion was made in this field by Cantor in the late nineteenth century, which lead to major inquiry throughout the first half of the twentieth century.

- We discuss basic ideas and notation here, which have much in common with the material from submodule 6.1

Set Membership

- **Sets are collections of elements. We denote that an object x is in a set A by writing $x \in A$.**

- **We denote that an object x is not in a set A by writing $x \notin A$.**

- **The set of things *not in a set* A is called the *complement of* A and is denoted A^c.**

- **If every element of a set B is also in the set A, we say**

$$B \text{ is a subset of } A, \text{ denoted } B \subset A.$$

- **If** $B \subset A$ but $A \neq B$, B is a proper subset of A

Set Algebra

- **One can discussion addition, subtraction and other operations on sets.**
- **The *union* of** $A, B, \text{ denoted } A \cup B$**, is the set of things in either** $A \text{ or } B.$
- **The *intersection* of** $A, B, \text{ denoted } A \cap B,$**is the set of things in both** $A \text{ and } B.$
- **The *difference* of the set** $A \text{ from } B$ **is the set of points in** $A \text{ not in } B.$
- **The set with *no elements* is called the *empty set*,** $\emptyset.$

$A = \{1, 4, 6, 7\}, B = \{-1, 3, 6, 7, 8\}$

Compute $A \cup B, A \cap B$

6.2.2 Set Theory II

Useful Rules for Set Algebra

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$
- $(A \cap A^c) = \emptyset$
- If $B \subset A, \text{ then } A \cup B = A, A \cap B = B.$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$

  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$

$$A = \{x \mid \text{such that } x \geq 3\} = [3, \infty)$$
$$B = \{x \mid \text{such that } x \leq 5\} = (-\infty, 5)$$

$$A \cup B$$

$$A \cap B$$

$$(\sim A) \cup B$$

$$A \cup (\sim B)$$

$$(\sim A) \cup (\sim B)$$

$$(\sim A) \cap B$$

$$A \cap (\sim B)$$

$$(\sim A) \cap (\sim B)$$

Cartesian Product

- **A related set theoretic notion that may be tested on the CLEP is that of *Cartesian product*.**

$$A \times B = \{(a, b) \text{ such that } a \in A, b \in B\}$$

- **In other words, a Cartesian product of two sets is pairs of elements, with the first element in the first set, the second element in the second set.**

- **Order matters, so unlike the normal notion of product,**

$$A \times B \neq B \times A.$$

$$A = \{1, 4, 5\}, B = \{2, 4, 7\}$$

State whether the following are true or false:

$$A \cap B = \emptyset$$

$$(1, 1) \in A \times B$$

$$(4, 4) \in A \times B$$

$$(1, 7) \in A \times B$$

$$(7,1) \in A \times B$$

$$(4,7) \in (A \cap B) \times B$$

7. Numbers

7.1 Properties of Numbers

7.2 Elementary Number Theory

7.3 Scientific Notation and Unit Conversion

7.4 Absolute Value

7.1 Properties of Numbers

Properties of Numbers

- For us, *real* numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers, though real numbers may be understood as a proper subset of the complex numbers.

- There are many subsets of real numbers that are familiar to us; we want to understand their properties.

- Most quantities used to describe things in the world may be understood as real numbers.

Integers

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

Rational Numbers

- **Rational numbers are *ratios/fractions* of integers.**
- **Any number of the form $\frac{p}{q}$ for p, q, integers is rational.**
- **In particular, every integer is also considered a rational number.**
- **One must take care: $q = 0$ is not permitted, as this involves division by 0.**
- **Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.**

Irrational Numbers

- **There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q.**

- **Such numbers are called *irrational*; there are many of them.**

- **Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$.**

- **These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat.***

Identify each of the following numbers as integer, rational, or irrational

$2 + \sqrt{2}$

$\frac{12}{6}$

$\frac{121}{11}$

π

$\frac{\pi}{3\pi}$

$-\frac{16}{3}$

$\frac{17}{4}$

$\sqrt{49} + \sqrt{64}$

e^0

Label as true or false:

- Every rational number is an integer.
- There are integers that are irrational.
- Every integer is rational.

Algebraic numbers are those that are roots of polynomials with integer coefficients. Give an example of a number that is irrational but algebraic.

7.2 Elementary Number Theory

Elementary Number Theory

- Number theory concerns properties of integers.
- It is one of the oldest mathematical subjects, and many of its most famous unsolved questions are rather easy to state.
- Indeed, number theory is well-known amongst mathematicians for having very hard answers to very simple-to-state question.

Divisibility

- We say an integer p *divides* an integer q if $q = p \times r$ for some integer r .
- Numbers that divide q are called the *divisors* of q .
- For any integer p , it is a simple exercise to show that both $1 \text{ and } p$ are divisors of p. If these are the only divisors, we call p a prime number.
- Primes are quite mysterious. There ancient mathematical problems related to them that are still unsolved.

Find the greatest common divisor of the following pairs:

$(20, 42)$

$(30, 36)$

$(9, 72)$

$(8, 24)$

Even and Odd Integers

- **The integers $\ldots, -4, -2, 0, 2, 4, \ldots$ are the *even* numbers.**
- **The remaining integers $\ldots, -3, -1, 1, 3, \ldots$ are the *odd* integers.**
- **One can define these in terms of divisors.**
- **The even numbers are the integers having 2 as a divisor.**
- **The odd numbers are the integers that do not have 2 as a divisor.**

Show that the sum of any two odd integers is even.

Show that the product of any two odd integers is odd.

Fundamental Theorem of Arithmetic

- **Sometimes called the Unique Prime Factorization Theorem.**

- **It is a classic result that has ancient roots.**

Every integer n may be written uniquely as

$$n = p_1^{m_1} p_2^{m_2} \ldots, p_r^{m_r}$$

for prime numbers $p_1, \ldots, p_r$

and positive integers $m_1, \ldots, m_r$

Find the prime factorization of the following integers:

8	17	100
15	25	2

7.3 Scientific Notation and Unit Conversion

Scientific Notation and Unit Conversion

- **This submodule addresses important ideas for scientific computation.**

- ***Scientific notation*** **is simply an efficient way of writing numbers with many zeros.**

- **For very large or very small numbers, it is more compact than traditional notation.**

- A number is written in scientific notation by writing it as

$$x = a \times 10^n$$

- Here, a is between 1 and 10 .
- If $x > 10,$ then $n \geq 1.$
- If $0 < x < 1,$ then $n \leq -1.$
- The exponent n counts the number of zeroes before the decimal point if positive, and the number of zeroes after the decimal point if negative.
- One may do algebra in scientific notation, by following the usual product and exponent rules.

Convert to scientific notation:

$.0007$

-74561

2310

$-.003$

Unit Conversion

- **Many applications of mathematics require shifting between two different measurement systems, such as feet and meters, gallons and liters, Fahrenheit and Celsius.**

- **The rule for conversion is simple:**

$$\text{Unit A} \xrightarrow{\times \frac{\text{Unit B}}{\text{Unit A}}} \text{Unit B}$$

Convert units:

$$100\,\frac{\text{km}}{\text{hour}} \ \text{to} \ \frac{\text{miles}}{\text{minute}}$$

$$5\,\frac{\text{US dollars}}{\text{gallon}} \text{ to } \frac{\text{Euros}}{\text{liter}}$$

$$12\,\frac{\text{feet}}{\text{second}} \text{ to } \frac{\text{inches}}{\text{minute}}$$

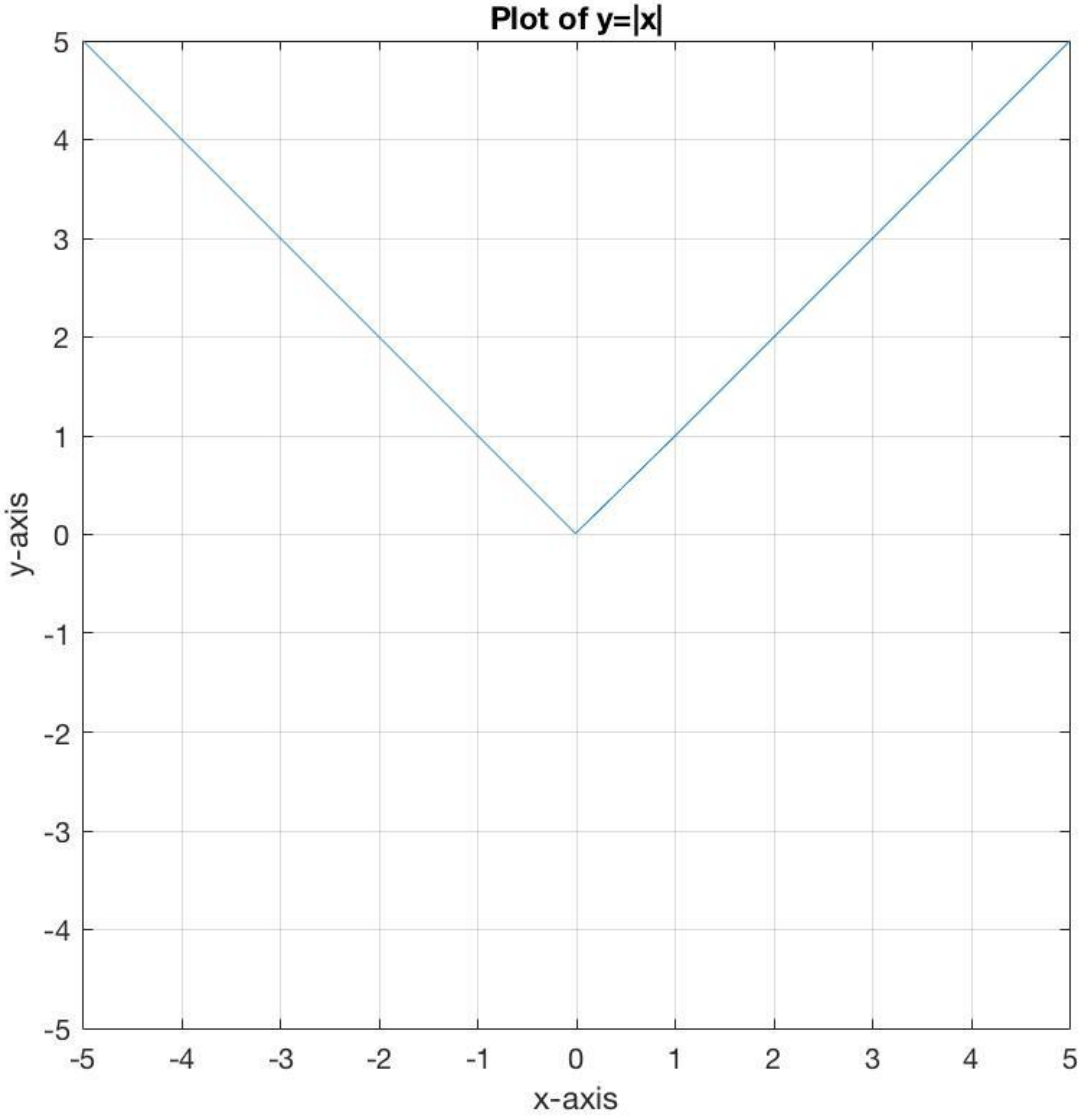
7.4 Absolute Value

Absolute Value

Recall the absolute value function, which is equal to a number's distance from 0:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, the absolute value function keeps positive numbers the same, and switches negative numbers into their positive counterpart.


Plot of y=|x|
y-axis
x-axis
5
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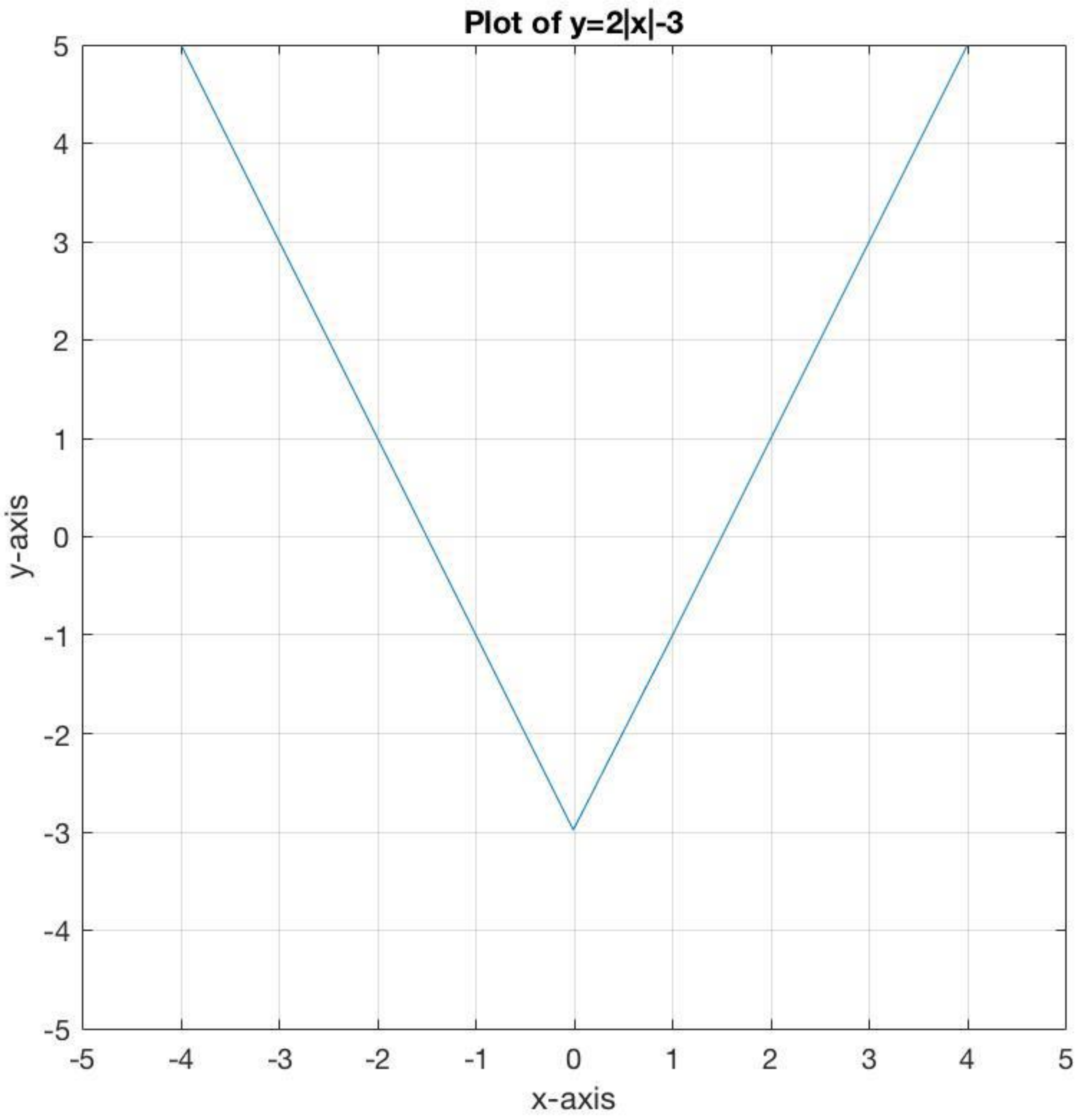
Plot of y=2|x|-3
y-axis
x-axis
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-5
-4
-3
-2
-1
0
1
2
3
4
5

Evaluate $f(2)$ if f has the following formulae:

$f(x) = |x + 1|$

$f(x) = -|x - 3|$

$f(x) = |x - 7|$

$f(x) = |x - 2|$

Equations with Absolute Value

When considering equations of the form: $|f(x)| = g(x)$ **,**

it suffices to consider the two cases

$$f(x) = g(x) \text{ and } -f(x) = g(x)$$

In the case of absolute value equations involving first order polynomials (linear functions), we get:

$$|ax+b| = c \Leftrightarrow ax+b = c \text{ or } -(ax+b) = c$$

Solve $|2x - 3| = 1$

Inequalities Involving Absolute Values

When considering systems of absolute value *inequalities*, great care must be taken.

In general,

$$|f(x)| \leq g(x) \Leftrightarrow \begin{cases} f(x) \leq g(x) \text{ and } f(x) > 0 \\ -f(x) \leq g(x) \text{ and } f(x) \leq 0 \end{cases}$$

A similar equivalence holds for $|f(x)| \geq g(x)$.

Linear Absolute Value Inequalities

- **One can, when working with inequalities of the form**

$$|ax + b| \leq c \text{ or } |ax + b| \geq c$$

proceed by finding the two solutions to

$$ax + b = c \text{ and } -(ax + b) = c$$

- **These can then be plotted on a number line, and checking in which region the desired inequality is achieved. This is the *number line method.***

Solve $|x+3| \leq 2$