2.1.1 Linear Equations and Inequalities

Equations of the form $y = ax + b$

We want to compute values of x given y and vice versa.

Sometimes we need to perform some algebraic rearrangements first

2.1.2 Linear Inequalities

Linear equations can be broadened to linear ***inequalities*** of the form $y \leq mx + b$, with $\geq, <, >$ potentially in place of $\leq$

Since $y = mx + b$ defines a line in the Cartesian plane, linear inequalities refer to all points on one side of a line, either including ($\leq, \geq$) or excluding ($<, >$) the line itself.

2.2.1 Quadratic Equations

Quadratic refers to degree two polynomials. Quadratic equations are equations involving degree two polynomials:

$$y = ax^2 + bx + c$$

Unlike linear equations, in which simple algebraic techniques were sufficient, finding solutions to quadratics requires more sophisticated techniques, such as:

- **Factoring**
- **Quadratic Formula**
- **Completing the Square**

2.2.2 Quadratic Formula

A formulaic approach to solving quadratic equations is the *quadratic formula:*

$$0 = ax^2 + bx + c \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, quadratic equations have two distinct roots, unless $b^2 - 4ac = 0$.

2.2.3 Quadratic Inequalities

$$ax^2 + bx + c \geq 0$$

Solving quadratic inequalities can be made easier with the observation that

$$AB \geq 0 \Leftrightarrow A \geq 0 \text{ and } B \geq 0$$
$$\text{or } A \leq 0 \text{ and } B \leq 0$$

This suggests factoring our quadratic, and examining when each linear factor is positive or negative.

Similarly,

$$AB \leq 0 \Leftrightarrow A \geq 0 \text{ and } B \leq 0$$
$$\text{or } A \leq 0 \text{ and } B \geq 0$$

Again, we see that if we can factor our quadratic into linear factors, we can examine each factor individually.

Indeed, supposing that our quadratic inequality has the form

$$x^2 + bx + c \geq 0,$$

we can factor $x^2 + bx + c = (x - \alpha)(x - \beta)$ and examine the corresponding linear factors.

2.3.1 Exponential and Logarithmic Equations

These may look daunting! However, we can use our exponential and logarithmic properties (tricks) to make our lives easier; see Lecture 1.3,1.4.

Recall that $y = a^x \Leftrightarrow \log_a(y) = x$.

From this, we can approach many equations that look intimidating.

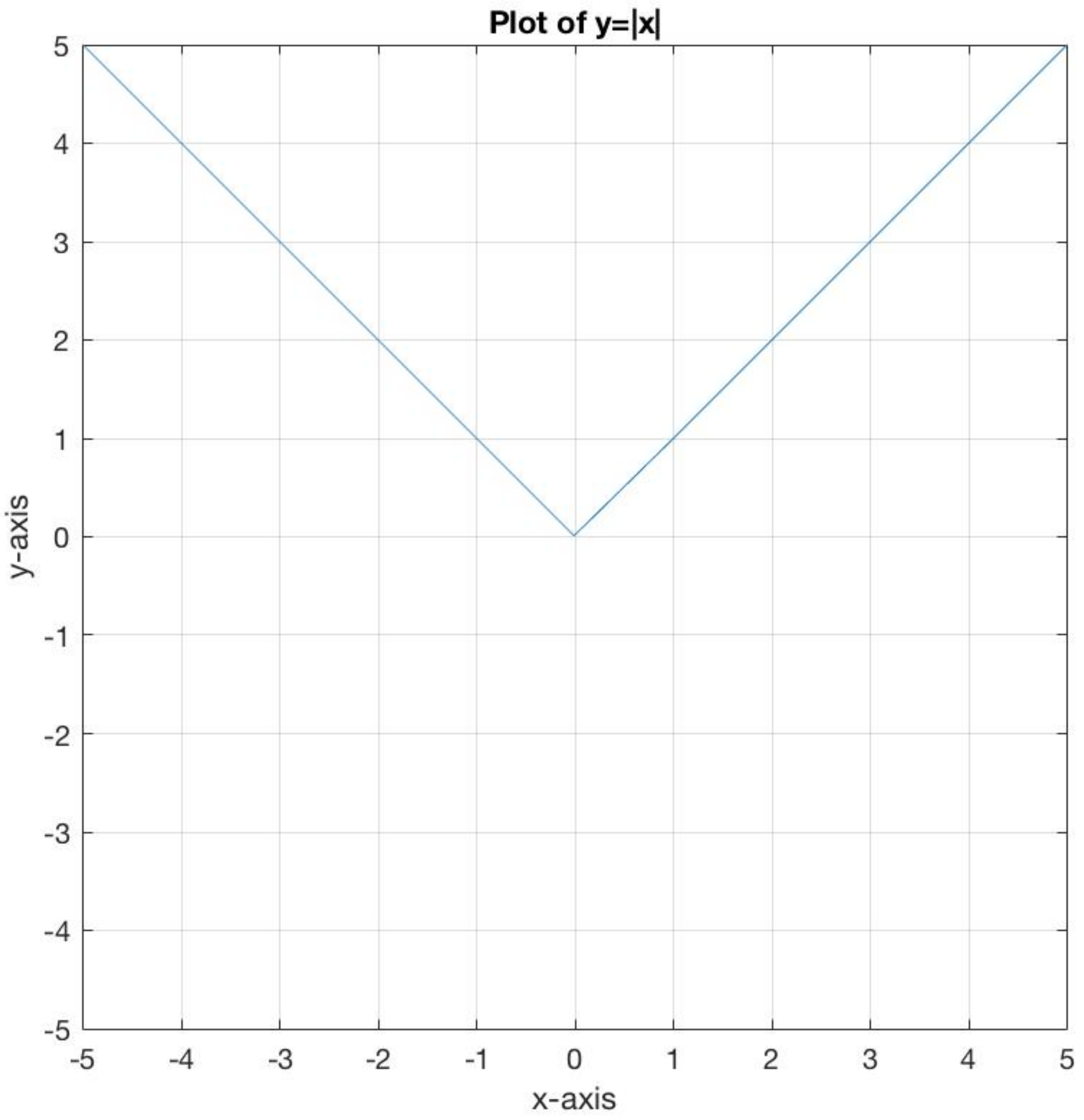
2.4.1 Absolute Value Equations

2.4.1 Absolute Value Equations

Recall the absolute value function, which is equal to a number's distance from 0:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, the absolute value function keeps positive numbers the same, and switches negative numbers into their positive counterpart.


Plot of y=|x|
5
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3
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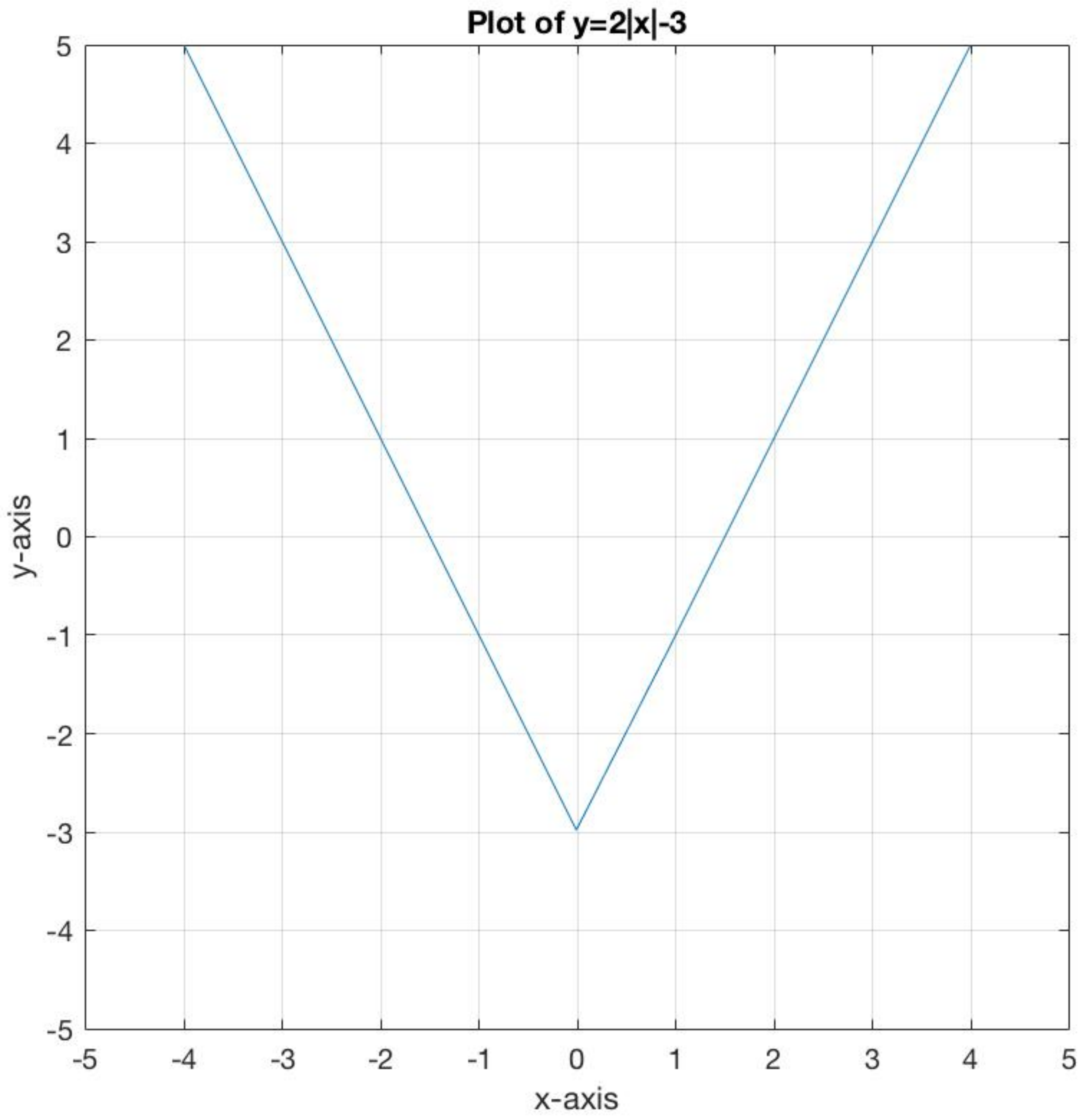
x-axis
y-axis


Plot of y=2|x|-3
y-axis
x-axis
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-5
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-2
-1
0
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2
3
4
5

2.4.2 Equations with Absolute Values

2.4.2 Equations with Absolute Value

When considering equations of the form: $|f(x)| = g(x)$ **,**

it suffices to consider the two cases

$$f(x) = g(x) \text{ and } -f(x) = g(x)$$

In the case of absolute value equations involving first order polynomials (linear functions), we get:

$$|ax + b| = c \Leftrightarrow ax + b = c \text{ or } -(ax + b) = c$$

2.4.3 Inequalities Involving Absolute Value

2.4.4 Linear Absolute Value Inequalities

One can, when working with inequalities of the form

$$|ax + b| \leq c \text{ or } |ax + b| \geq c$$

proceed by finding the two solutions to

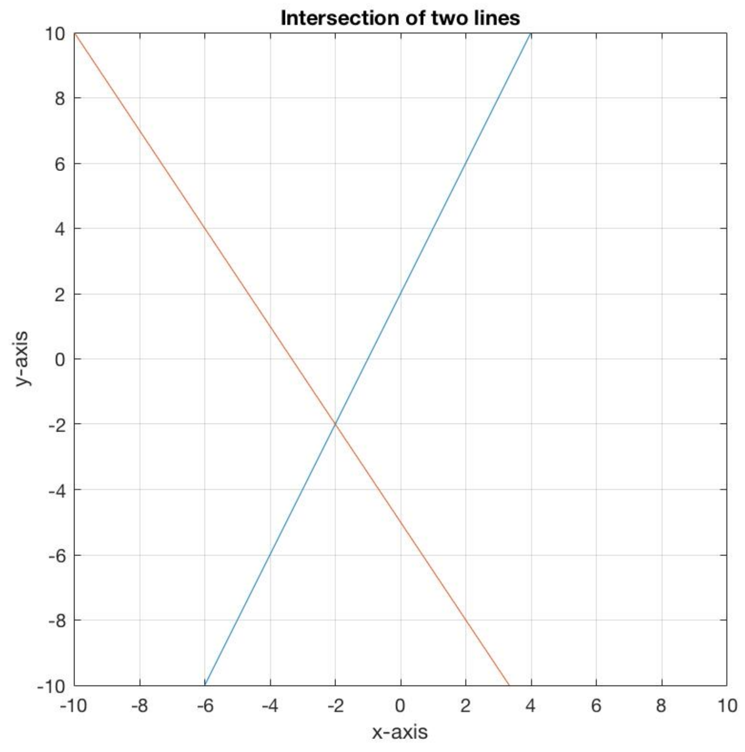
$$ax + b = c \text{ and } -(ax + b) = c$$

then plotting these on a number line, and checking in which region the desired inequality is achieved. This is the *number line method.*

2.5.1 Systems of Equations and Inequalities

A classic area of mathematics is solving two or more systems of equations or inequalities *simultaneously.*

On classic formulation is: find the intersection of two lines, given their equations



2.5.2 Systems of Linear Equations

The problem of finding the intersection of two lines may formulated as the algebraic problem of finding the simultaneous solution to system of linear equations

$$\begin{cases} y = m_1x + b_1 \\ y = m_2x + b_2 \end{cases}$$

Classical solution method: Set the two expressions on the right equal and solve for x, then go back and solve for y.

2.5.3 Higher Order Systems

It is possible to mix other types of equations into systems. The same techniques as before work.

$$\begin{cases} y = 3x + 4 \\ y = x^2 + x + 1 \end{cases}$$

While more complicated looking, this system can be solved with our substitution method, combined with the *quadratic formula*.

2.5.4 Systems of Inequalities

One can also study regions in the Cartesian plane in which a inequalities are simultaneously satisfied.

In the case of linear inequalities, these may be of the form:

$$\begin{cases} y \leq m_1 x + b_1 \\ y \leq m_2 x + b_2 \end{cases} \qquad \begin{cases} y \geq m_1 x + b_1 \\ y \leq m_2 x + b_2 \end{cases} \qquad \begin{cases} y \geq m_1 x + b_1 \\ y \geq m_2 x + b_2 \end{cases}$$