1.1.1 Algebraic Operations

We need to learn how our basic algebraic operations interact.

When confronted with many operations, we follow *the order of operations:*

Parentheses 1
Exponentials 2
Multiplication 3
Division 4
Addition 5
Subtraction 6

ex: Simplify $x^2(x+3) - (x\cdot 2x+1)$

$$= x^2\cdot(x+3) - (2x^2+1)$$

$$= x^2\cdot x + x^2\cdot 3 - (2x^2+1)$$

$$= x^3 + 3x^2 - (2x^2+1)$$

$$= x^3 + \underline{3x^2} - \underline{2x^2} - 1$$

$$= x^3 + x^2 - 1$$

ex: $\left(3 \cdot \left(7^2 \div 49\right) + 2\right) \div 5$

$= \left(3 \cdot \left(49 \div 49\right) + 2\right) \div 5$

$= \left(3 \cdot (1) + 2\right) \div 5$

$= (3 + 2) \div 5$

$= 5 \div 5$

$= 1$ ✓

1.1.2 Manipulating Fractions

Fractions are essential to mathematics

- **Adding fractions:** $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
- **Multiplying fractions:** $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
- **Improper fractions:** $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

ex: Simplify $\frac{a^5 b^2}{a^3 b}$

ex: Compute $\frac{1}{3} + \frac{7}{5} + \frac{2}{7}$

1.1.3 Powers and Roots

For any positive whole number $n, x^n = \underbrace{x \cdot x \cdot ... \cdot x}_{\text{n times}}$

If $y = x^n$**, we say** x **is an** n^{th} **root of** y**.**

Roots undo powers, and vice versa. We denote n^{th} **roots as** $\sqrt[n]{x}$ **or** $x^{\frac{1}{n}}$ **.**

Notice that:

$$\left(x^{\frac{1}{n}}\right)^n$$

$$=x^{\frac{1}{n}\cdot n}$$

$$=x^1$$

$$=x$$

ex: Simplify $\dfrac{\left(\sqrt[4]{9}\right)^8}{4^2 - \sqrt[3]{8}}$

1.2.1 Factoring and Expanding Polynomials

- **Polynomials are functions that are sums of nonnegative integer powers of the variables.**
- **The highest power is called the degree of the polynomial.**
- **Higher degree polynomials are generally harder to understand.**

$$y = x^2 \qquad f(x) = x - 1 \qquad f(r) = 4r^2 - 9$$

$$y = x^2 + 2x - 1 \qquad f(x) = x^6 - x + 1$$

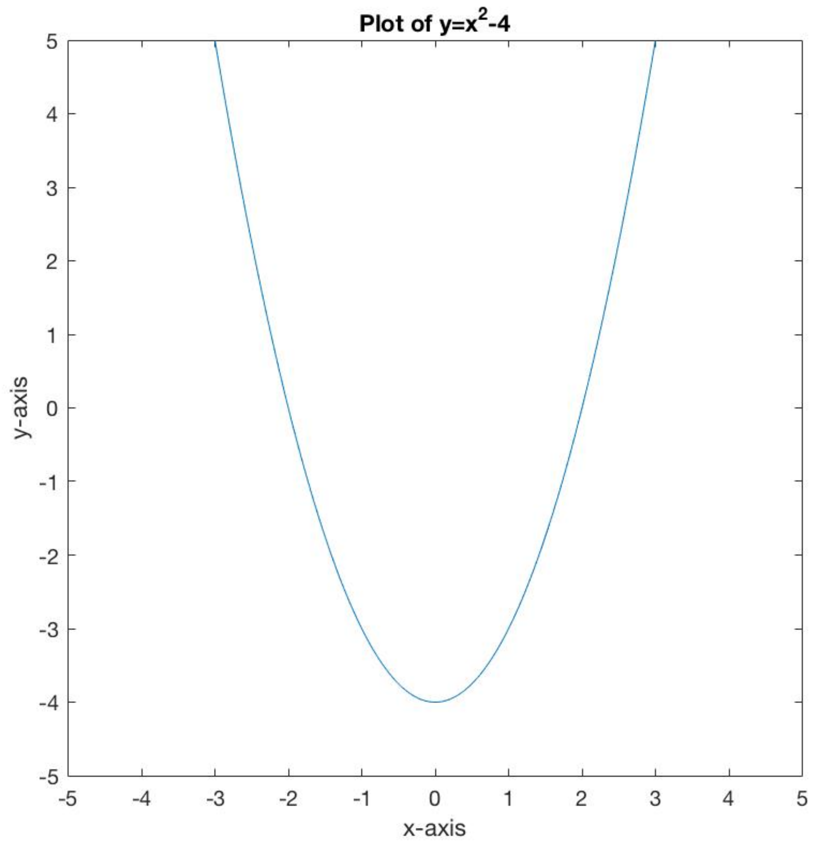
1.2.2 First Order Polynomials

These are just lines:

$$y = ax + b$$

1.2.3 Second Degree Polynomials

$$y = ax^2 + bx + c$$



We can try to **factor** quadratics, i.e. write as a product of first order polynomials

1.2.4 Roots of Quadratics

These can be found by factoring, and also with the famous *quadratic formula:*

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.2.5 Higher Order Polynomials

- These can also be factored, though it is usually harder.
- Formulas like the quadratic formula exist for degree 3,4, polynomials.
- Nothing for degree 5 and higher.

1.2.6 Expanding Polynomials

One can undo factoring by expanding products of polynomials. One must take care with distribution.

$$(ax+b)(cx+d) \neq acx^2 + bd$$

$$(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$$

1.3.1 Introduction to Exponentials

x^y y

- The number x is called the base.
- The number y is called the exponent.

- Examples include: x^2 (base x, exponent 2) and 2^x (base 2, exponent x)

- When someone refers to an *exponential function*, they mean the variable is in the exponent (i.e. 2^x), not the base (x^2)

ex: $4^{(x^2)}$

base = 4

exponent = x^2

1.3.2 Properties of Exponents

Basic Rules:

- **(same base, different exponents)**

$$a^{x+y} = a^x a^y$$

- **(different base, same exponent)**

$$a^x b^x = (ab)^x$$

- **(iterated exponents)**

$$(a^x)^y = a^{xy}$$

- **(for any value of x, by convention)**

$$x^0 = 1$$

already seen with degree 0 polynomials...

ex: $2^{2^2} \rightarrow 2 \cdot 2$

$2^{(2 \cdot 2)}$

$= 2^{4}$

$= 16$

ex: Simplify

$$2^{-3} \cdot 2^{3}$$

$$= 2^{-3+3}$$

$$= 2^{0}$$

$$= 1$$

ex: Simplify

$$2^4 \cdot (2^{-1})^3$$

$$= 2^4 \cdot 2^{-1 \cdot 3}$$

$$= 2^4 \cdot 2^{-3}$$

$$= 2^{4+-3}$$

$$= 2^1 = 2 \checkmark$$

ex: Simplify

$\sqrt{3} \cdot \sqrt{27}$

$= 3^{1/2} \cdot 27^{1/2}$

$= (3 \cdot 27)^{1/2}$

$= (81)^{1/2}$

$= \sqrt{81} = 9$

$\sqrt{x} = x^{1/2}$

ex: Simplify

$$a^4 \cdot b^2 \cdot a^{-1/2} \cdot b^{-3}$$

$$= (a^4 \cdot a^{-1/2}) \cdot (b^2 \cdot b^{-3})$$

$$= (a^{4-1/2}) \cdot (b^{2-3})$$

$$= a^{7/2} \cdot b^{-1}$$

$$= a^{7/2} b^{-1}$$

ex:

Simplify $\underbrace{x^{1-y} \cdot x^{y+7} \cdot x^{-6}}_{\text{have base } x}$

$= x^{[(1-\cancel{y})+(\cancel{y}+7)+(-6)]}$

$= x^2$

ex : Simplify

$$\underbrace{3^x \cdot 2^x}_{} \cdot 6^{-x}$$

$$= (3 \cdot 2)^x \cdot 6^{-x}$$

$$= 6^x \cdot 6^{-x}$$

$$= 6^{x-x}$$

$$= 6^0 = 1$$

ex: Simplify $\dfrac{x^2 \cdot x^{-3}}{x^5} \cdot \dfrac{x^4}{x} \cdot x$

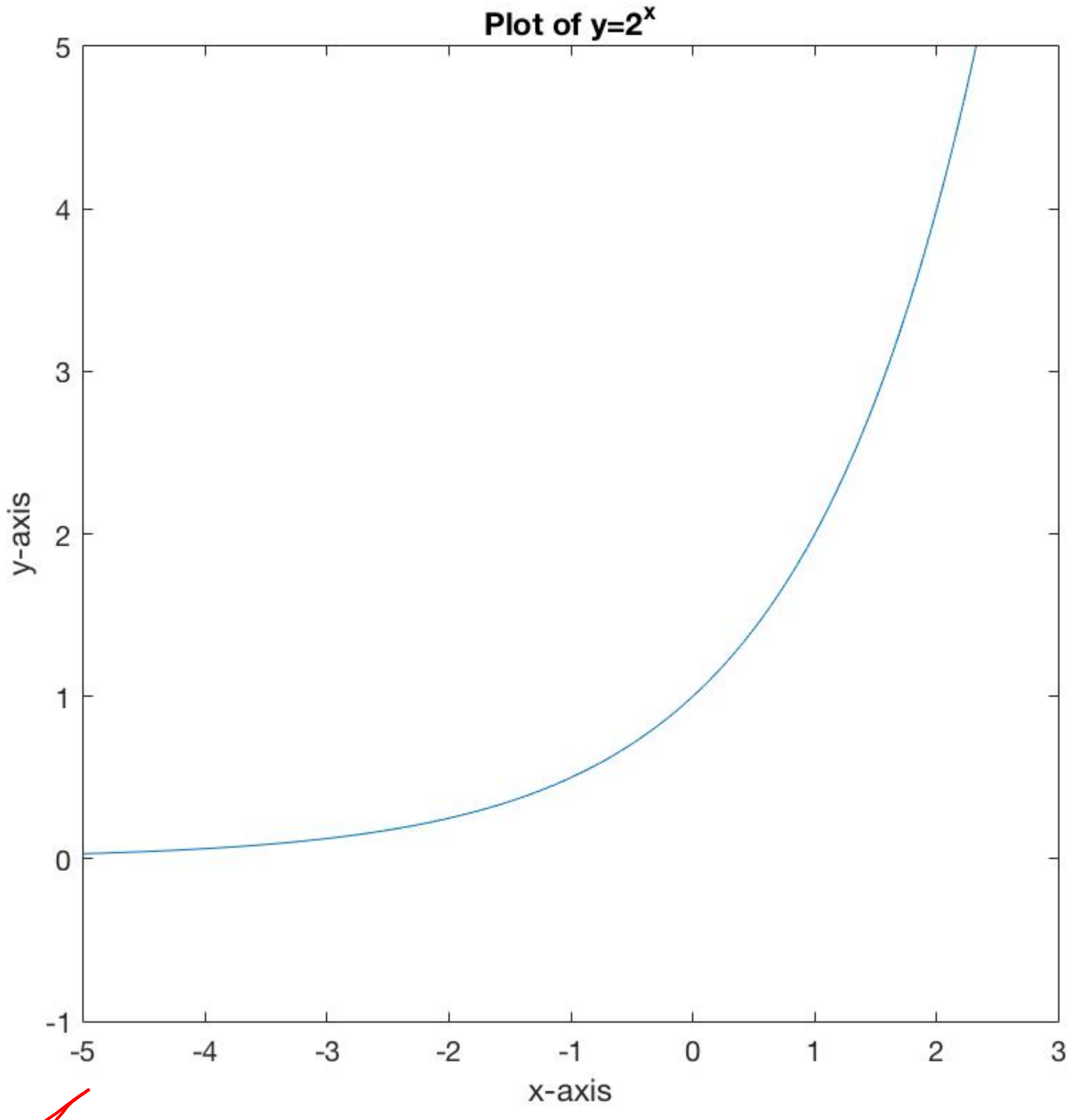
$\dfrac{1}{x} = x^{-1}$

$$= \frac{x^{2-3}}{x^5} \cdot \frac{x^4}{\cancel{x}} \cdot \cancel{x}$$

$$= \frac{x^{-1}}{x^5} \cdot x^4 = \frac{x^{-1} \cdot x^4}{x^5}$$

$$= \frac{x^3}{x^5} = x^{-2}$$

1.3.3 Plots of Exponentials

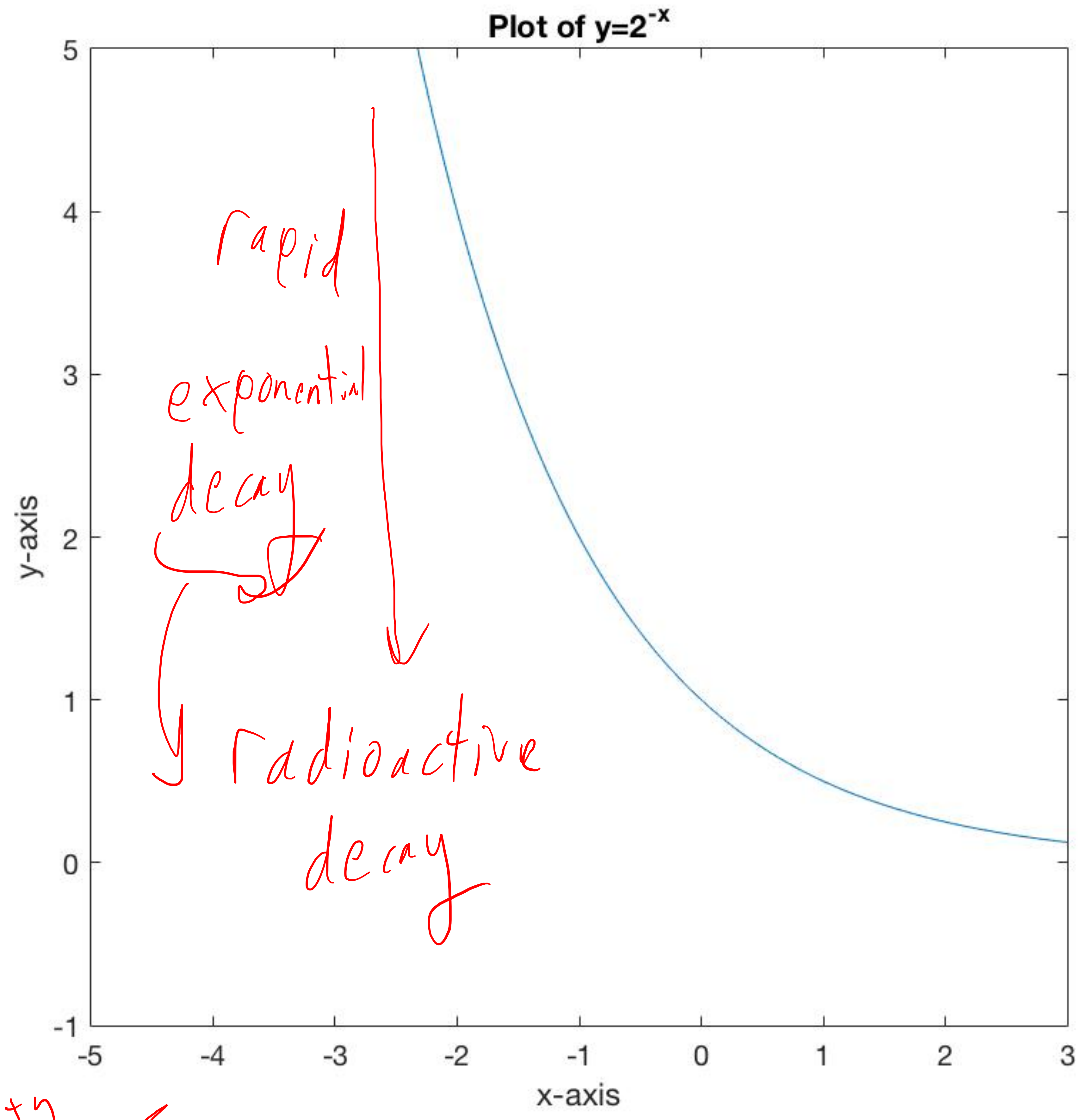


exponential growth

rapid growth

x	$y=2^x$
-1	1/2
0	1
1	2
2	4
3	8

x	$y = 2^{-x}$
-1	$2^{-(-1)} = 2$
0	1
1	$1/2$
2	$1/4$
3	$1/8$



growth $\longleftrightarrow$ decay

1.4.1 Logarithms

$$y = \log_a(x)$$

- We call a the base.
- Logarithms are a compact way to solve certain exponential equations:

$$y = \log_a(x) \Leftrightarrow a^y = x$$

1.4.2 Properties of Logarithms

Logarithms enjoy certain algebraic properties, related to the exponential properties we have already studied.

- $\log_a(xy) = \log_a(x) + \log_a(y)$ **(logarithm of a product)**
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ **(logarithm of a quotient)**
- $\log_a(x^y) = y\log_a(x)$ **(logarithm of an exponential)**
- $\log_a(1) = 0$ **(logarithm of 1 equals 0)**

analogous $a^0 = 1 \longleftrightarrow \log_a(1) = 0$

ex: $\log_{10}(x^2y^3)$

$= \log_{10}(x^2) + \log_{10}(y^3)$

$= 2\log_{10}(x) + 3\log_{10}(y)$ ✓

ex: $\log_2(16)$

$= \log_2(2^4)$ $\quad 16 = 2^4$

$= 4 \cdot \underbrace{\log_2(2)}$

$= 4 \cdot 1 = 4$ ✓

$a^1 = a$

$\Updownarrow$

$\log_a(a) = 1$, for any choice of a.

ex: $\log_4(2^x \cdot 2^x)$

$= \log_4((2 \cdot 2)^x)$

$= \log_4(4^x)$

$= x \cdot \log_4(4)$

$= x \cdot 1 = x$ ✓

ex: $\log_2\left(\frac{x^2}{8^y}\right)$

$$= \log_2(x^2) - \log_2(8^y)$$

$$= 2\cdot\log_2(x) - y\underbrace{\log_2(8)}_{=3}$$

$$= 2\log_2(x) - 3y$$

ex: $\log_{10}\left(\frac{x^2}{x^4} - x^{-2} + 1\right)$

Note: $\frac{\cancel{x^2}}{\cancel{x^4}} = x^{-2}$

$= \log_{10}\left(\cancel{x^{-2}} - \cancel{x^{-2}} + 1\right)$

$= \log_{10}(1)$

$= 0$ ✓

Fact $\log_a(1) = 0$, for every base a.

eg:

$3^{\log_3(4)}$

$= \log_3(3^4)$

$= 4$

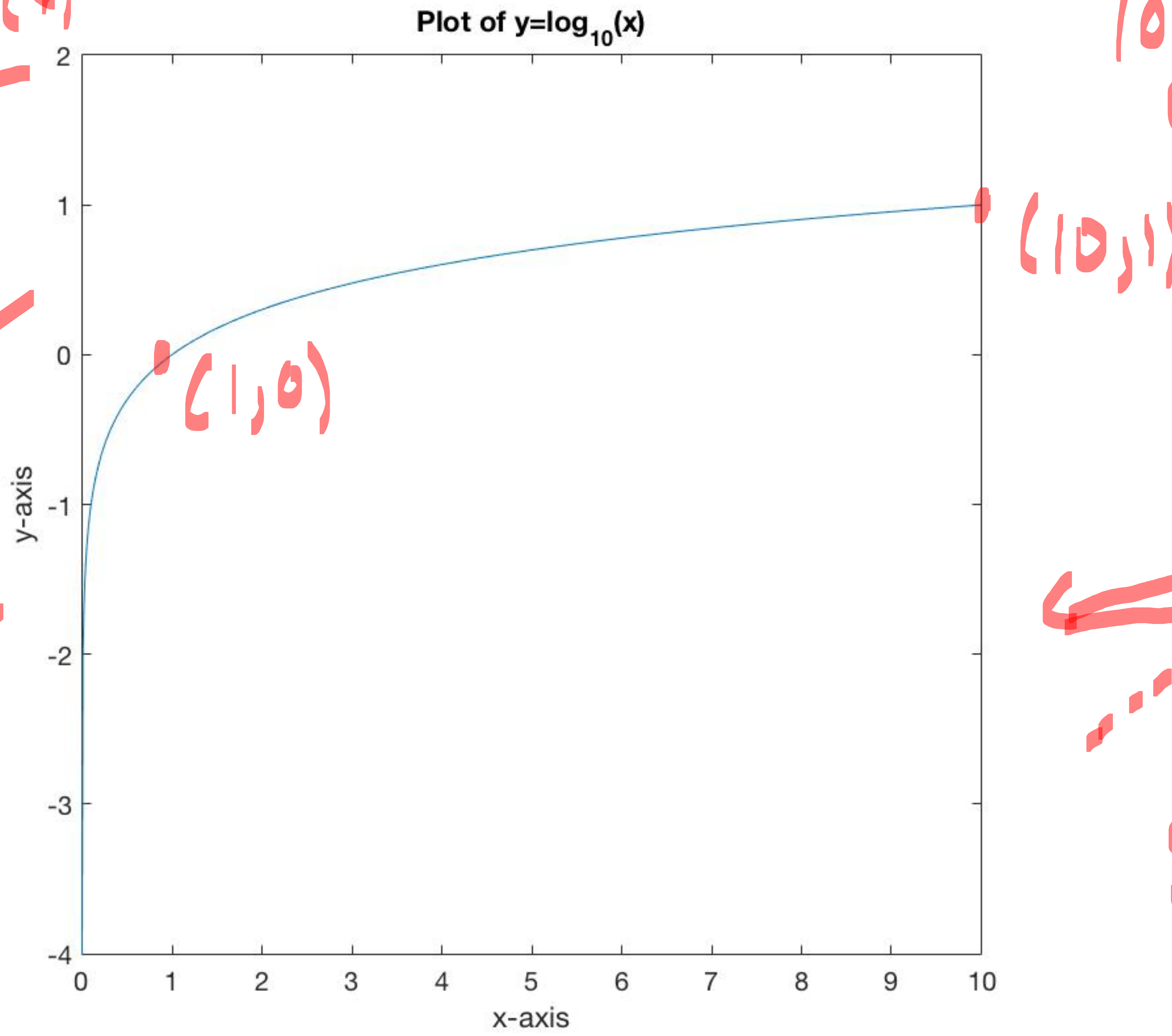
1.4.3 Logarithm as Inverse of Exponential

$$\log_a(a^x) = a^{\log_a(x)} = x$$

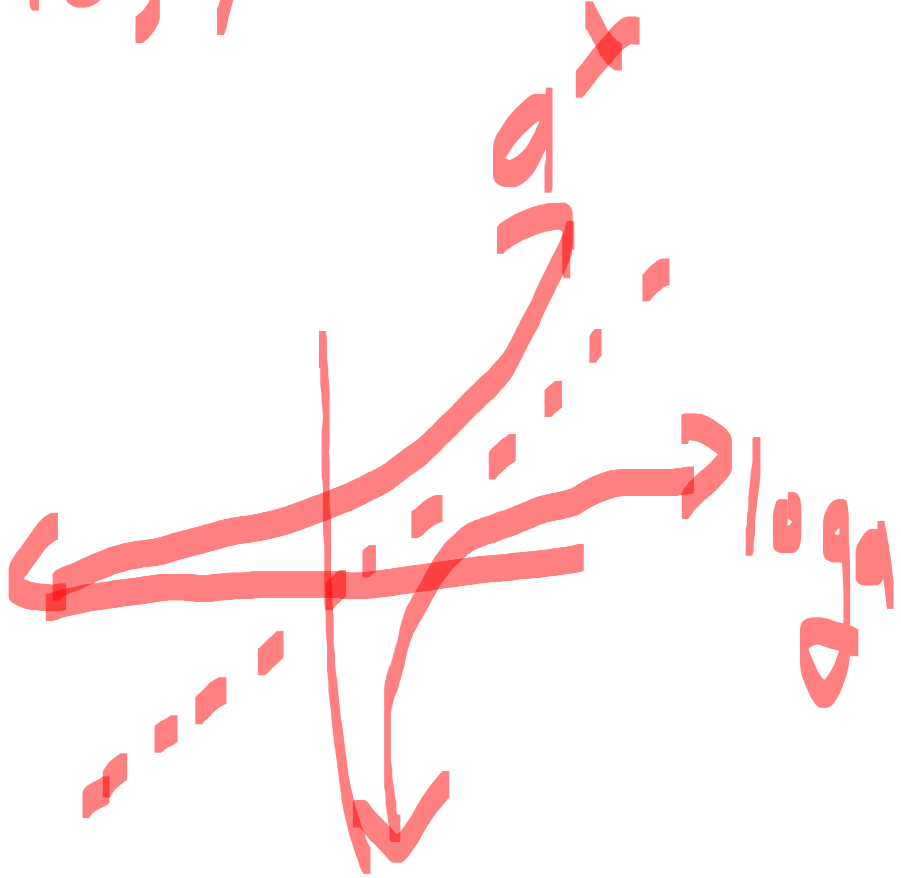
i.e. $\log_a$ undoes a^x, vice versa

x	$y=\log_{10}(x)$
1	0
10	1
100	2

we see logarithms grow very slowly



$\log_{10}(7) \approx ?$



Sketch

2.1.1 Linear Equations and Inequalities

Equations of the form $y = ax + b$

We want to compute values of x given y and vice versa.

Sometimes we need to perform some algebraic rearrangements first

2.1.2 Linear Inequalities

Linear equations can be broadened to linear ***inequalities*** of the form $y \leq mx + b$, with $\geq, <, >$ potentially in place of $\leq$

Since $y = mx + b$ defines a line in the Cartesian plane, linear inequalities refer to all points on one side of a line, either including ($\leq, \geq$) or excluding ($<, >$) the line itself.

2.2.1 Quadratic Equations

Quadratic refers to degree two polynomials. Quadratic equations are equations involving degree two polynomials:

$$y = ax^2 + bx + c$$

Unlike linear equations, in which simple algebraic techniques were sufficient, finding solutions to quadratics requires more sophisticated techniques, such as:

- Factoring
- Quadratic Formula
- Completing the Square

2.2.2 Quadratic Formula

A formulaic approach to solving quadratic equations is the *quadratic formula:*

$$0 = ax^2 + bx + c \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, quadratic equations have two distinct roots, unless $b^2 - 4ac = 0$.

2.2.3 Quadratic Inequalities

$$ax^2 + bx + c \geq 0$$

Solving quadratic inequalities can be made easier with the observation that

$$\begin{aligned} AB \geq 0 \Leftrightarrow & A \geq 0 \text{ and } B \geq 0 \\ & \text{or } A \leq 0 \text{ and } B \leq 0 \end{aligned}$$

This suggests factoring our quadratic, and examining when each linear factor is positive or negative.

Similarly,

$$AB \leq 0 \Leftrightarrow A \geq 0 \text{ and } B \leq 0$$
$$\text{or } A \leq 0 \text{ and } B \geq 0$$

Again, we see that if we can factor our quadratic into linear factors, we can examine each factor individually.

Indeed, supposing that our quadratic inequality has the form

$$x^2 + bx + c \geq 0,$$

we can factor $x^2 + bx + c = (x - \alpha)(x - \beta)$ **and examine the corresponding linear factors.**

2.3.1 Exponential and Logarithmic Equations

These may look daunting! However, we can use our exponential and logarithmic properties (tricks) to make our lives easier; see Lecture 1.3,1.4.

Recall that $y = a^x \Leftrightarrow \log_a(y) = x$.

From this, we can approach many equations that look intimidating.

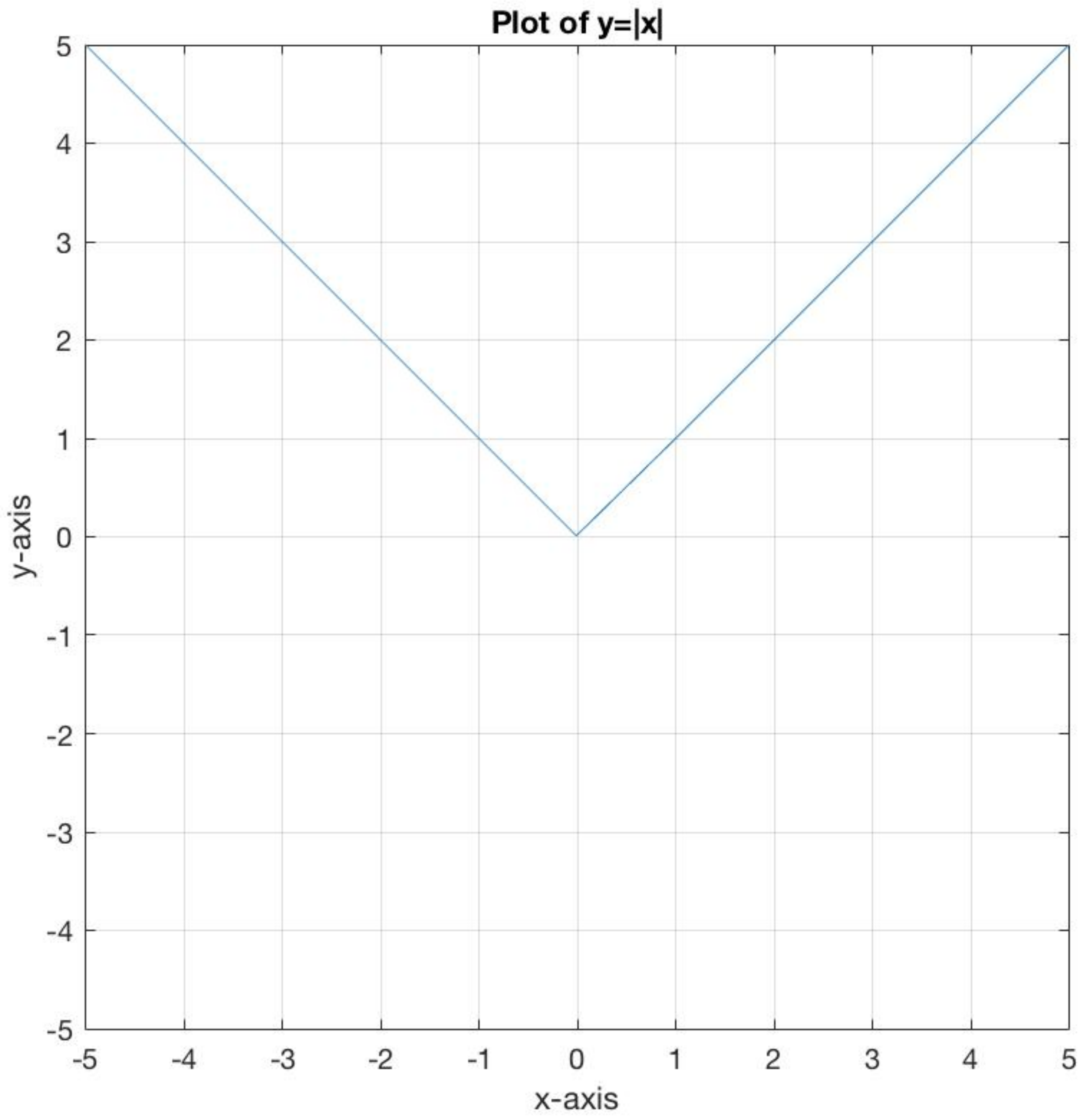
2.4.1 Absolute Value Equations

2.4.1 Absolute Value Equations

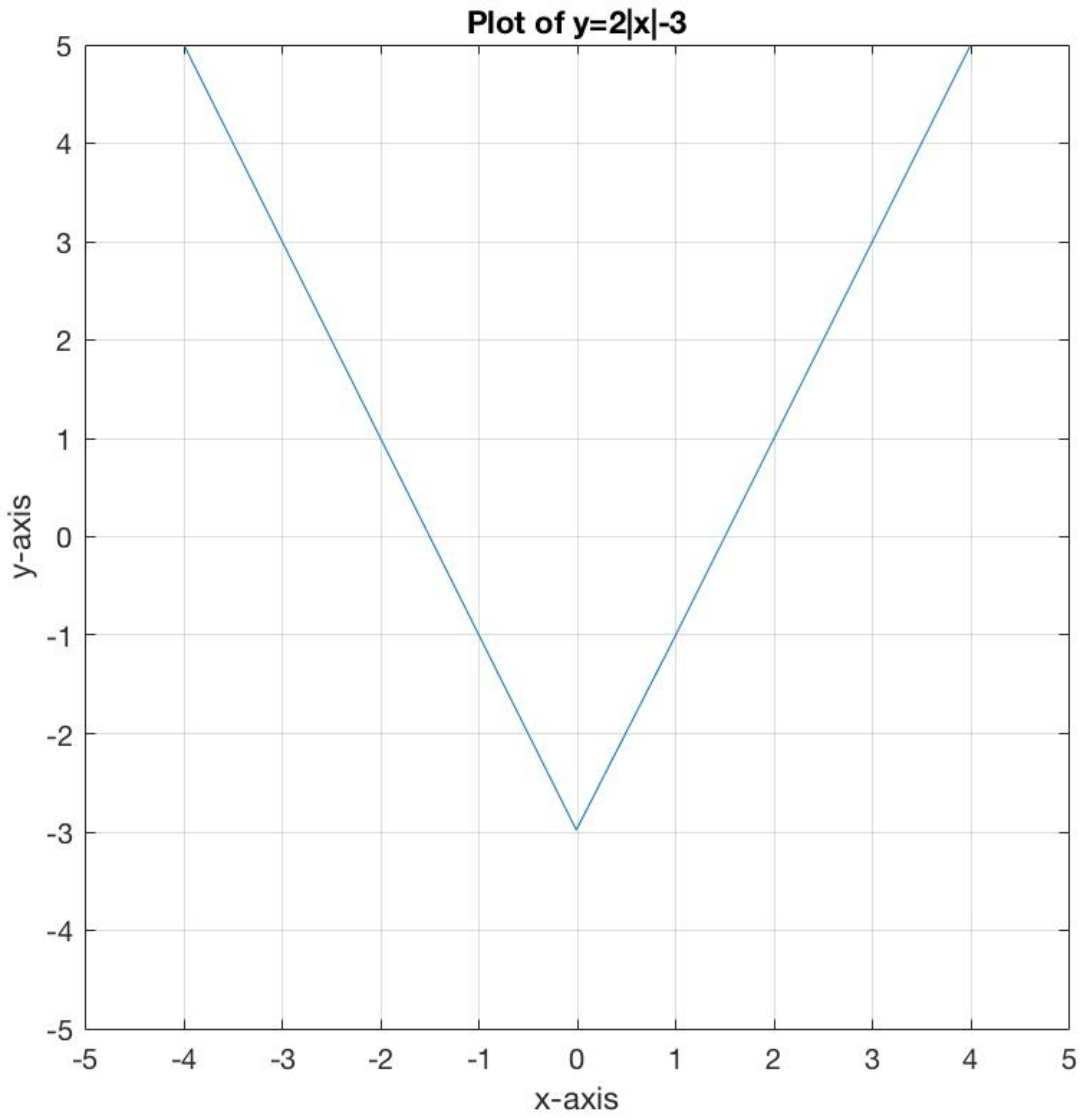
Recall the absolute value function, which is equal to a number's distance from 0:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, the absolute value function keeps positive numbers the same, and switches negative numbers into their positive counterpart.



Plot of y=|x|
5
4
3
2
1
0
-1
-2
-3
-4
-5
y-axis
-5
-4
-3
-2
-1
0
1
2
3
4
5
x-axis


Plot of y=2|x|-3
5
4
3
2
1
0
-1
-2
-3
-4
-5
y-axis
-5
-4
-3
-2
-1
0
1
2
3
4
5
x-axis

2.4.2 Equations with Absolute Values

2.4.2 Equations with Absolute Value

When considering equations of the form: $|f(x)| = g(x)$ **,**

it suffices to consider the two cases

$$f(x) = g(x) \text{ and } -f(x) = g(x)$$

In the case of absolute value equations involving first order polynomials (linear functions), we get:

$$|ax + b| = c \Leftrightarrow ax + b = c \text{ or } -(ax + b) = c$$

2.4.3 Inequalities Involving Absolute Value

2.4.4 Linear Absolute Value Inequalities

One can, when working with inequalities of the form

$$|ax + b| \leq c \text{ or } |ax + b| \geq c$$

proceed by finding the two solutions to

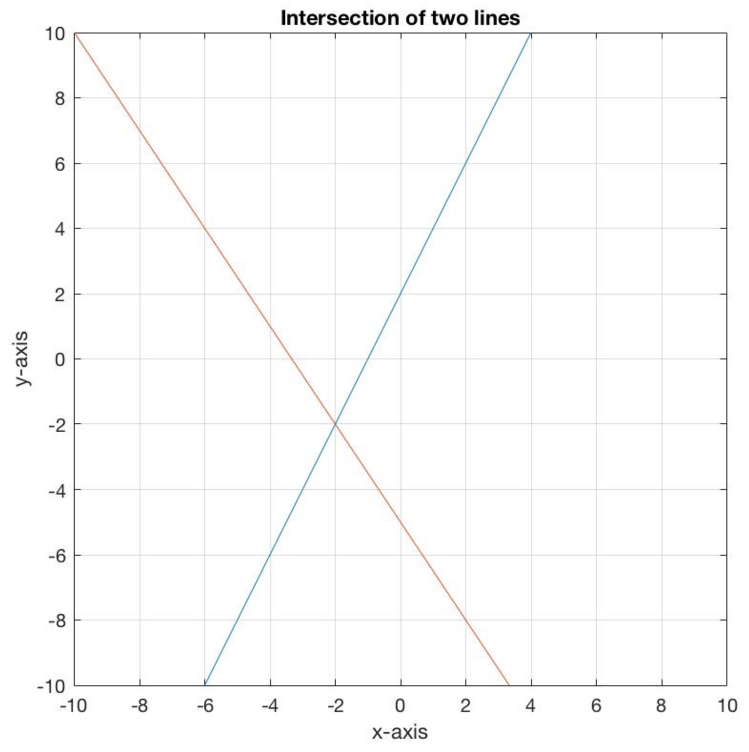
$$ax + b = c \text{ and } -(ax + b) = c$$

then plotting these on a number line, and checking in which region the desired inequality is achieved. This is the *number line method.*

2.5.1 Systems of Equations and Inequalities

A classic area of mathematics is solving two or more systems of equations or inequalities *simultaneously.*

On classic formulation is: find the intersection of two lines, given their equations



2.5.2 Systems of Linear Equations

The problem of finding the intersection of two lines may formulated as the algebraic problem of finding the simultaneous solution to system of linear equations

$$\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$$

Classical solution method: Set the two expressions on the right equal and solve for x, then go back and solve for y.

2.5.3 Higher Order Systems

It is possible to mix other types of equations into systems.
The same techniques as before work.

$$\begin{cases} y = 3x + 4 \\ y = x^2 + x + 1 \end{cases}$$

While more complicated looking, this system can be solved with our substitution method, combined with the *quadratic formula*.

2.5.4 Systems of Inequalities

One can also study regions in the Cartesian plane in which a inequalities are simultaneously satisfied.

In the case of linear inequalities, these may be of the form:

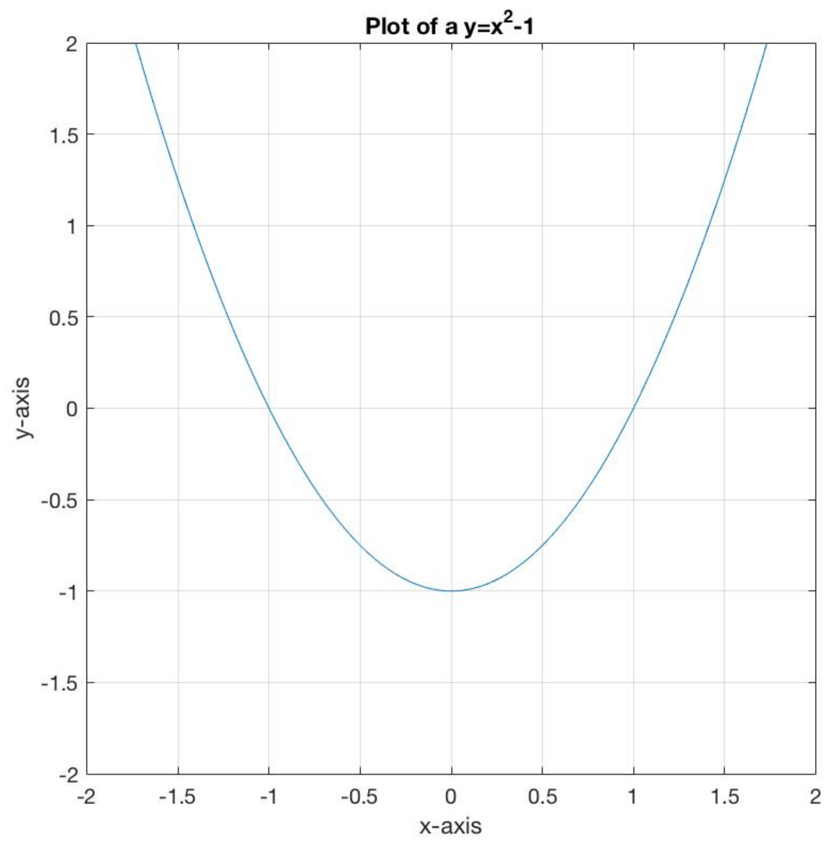
$$\begin{cases} y \leq m_1 x + b_1 \\ y \leq m_2 x + b_2 \end{cases} \qquad \begin{cases} y \geq m_1 x + b_1 \\ y \leq m_2 x + b_2 \end{cases} \qquad \begin{cases} y \geq m_1 x + b_1 \\ y \geq m_2 x + b_2 \end{cases}$$

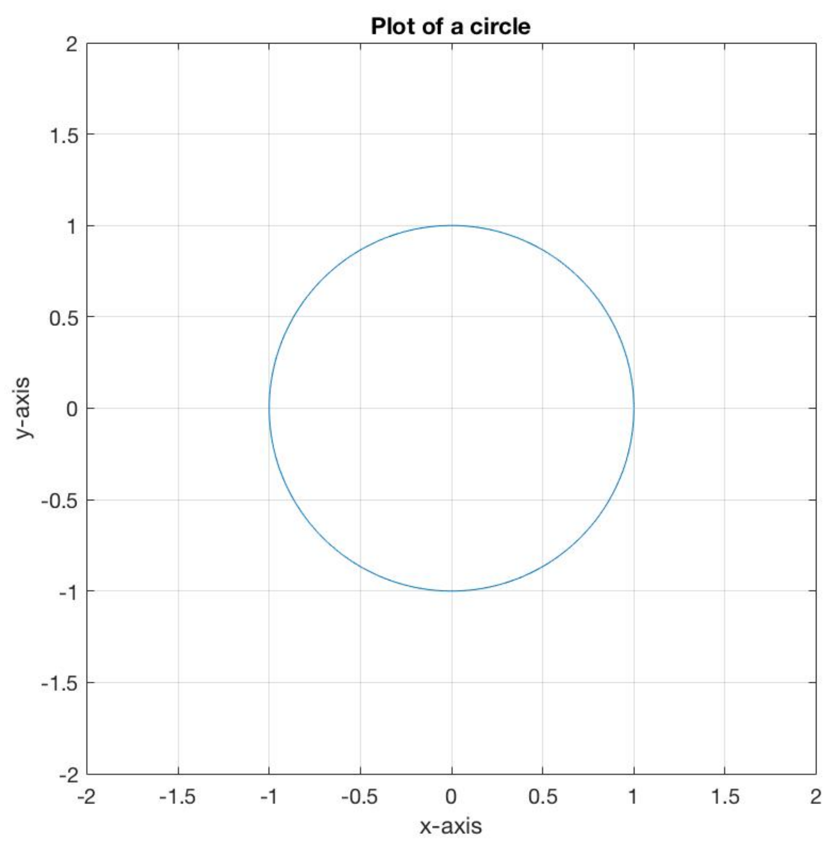
3.1.1 What is a Function?

- Functions are mathematical objects that send an input to a unique output.
- They are often, but not always, numerical.
- The classic notation is that $f(x)$ denotes the output of a function f at input value x.
- Functions are abstractions, but are very convenient for drawing mathematical relationships, and for analyzing these relationships.

3.1.2 Function or not?

One of the key properties of a function is that it assigns a unique output to an input.

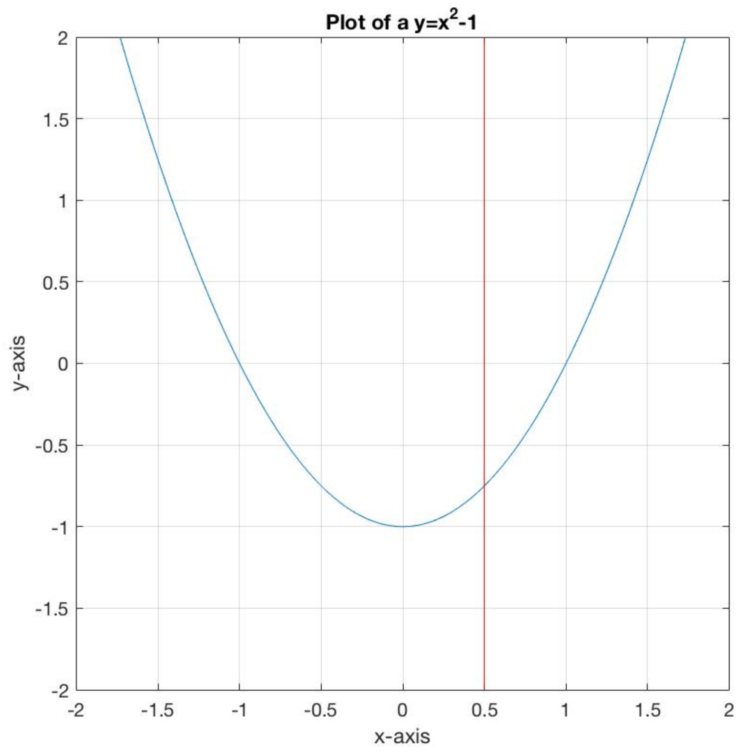


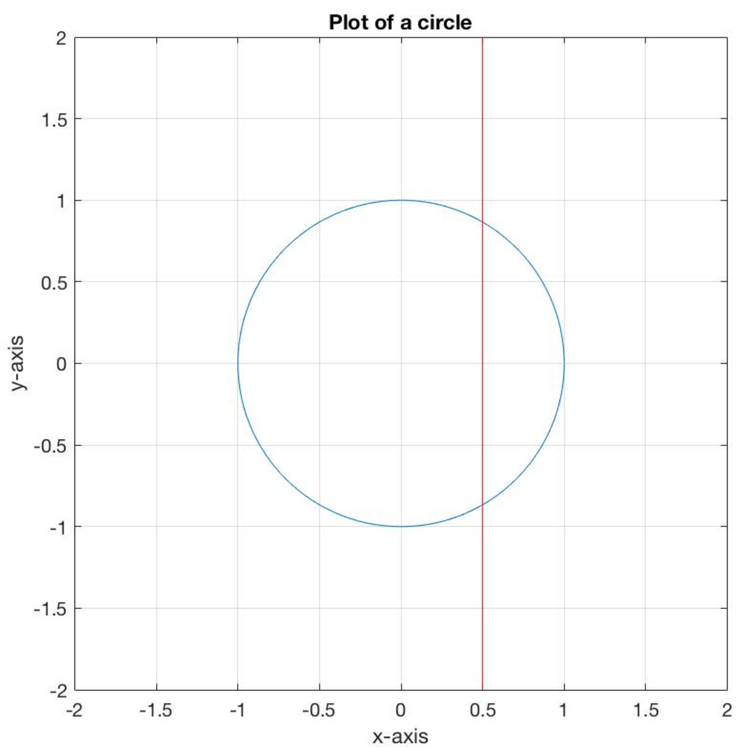


3.1.3 Vertical Line Test

A trick for checking if a mathematical relationship plotted in the Cartesian plane is a function is the *vertical line test.*

VLT: A plot is a function if and only if every vertical line intersects the plot in at most one place.





3.2.0 Representing with Functions

Functions are convenient for describing numerical relationships.

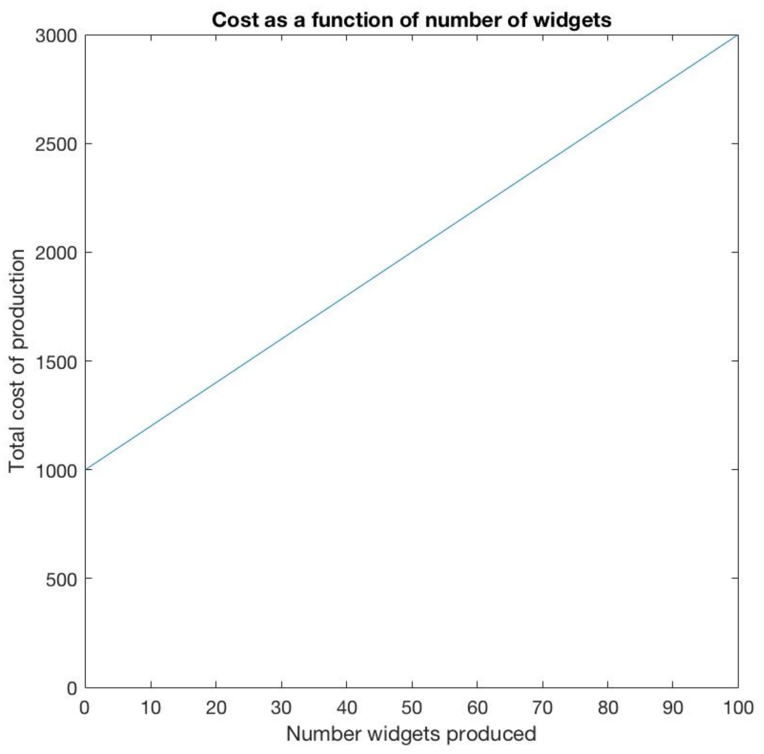
$$\text{Input} \xrightarrow{f} \text{Output}$$

To model a relationship with functions, you simply need to understand how your input depends on your output.

3.2.1 Linear Modeling

Some simple relationships can be modeled with *linear relationships* of the form $f(x) = ax + b$

For example, suppose the cost of producing a widgets is \$1000, plus \$20 for each widget produced. Then the total cost of producing x **widgets is modeling as** $C(x) = 20x + 1000$

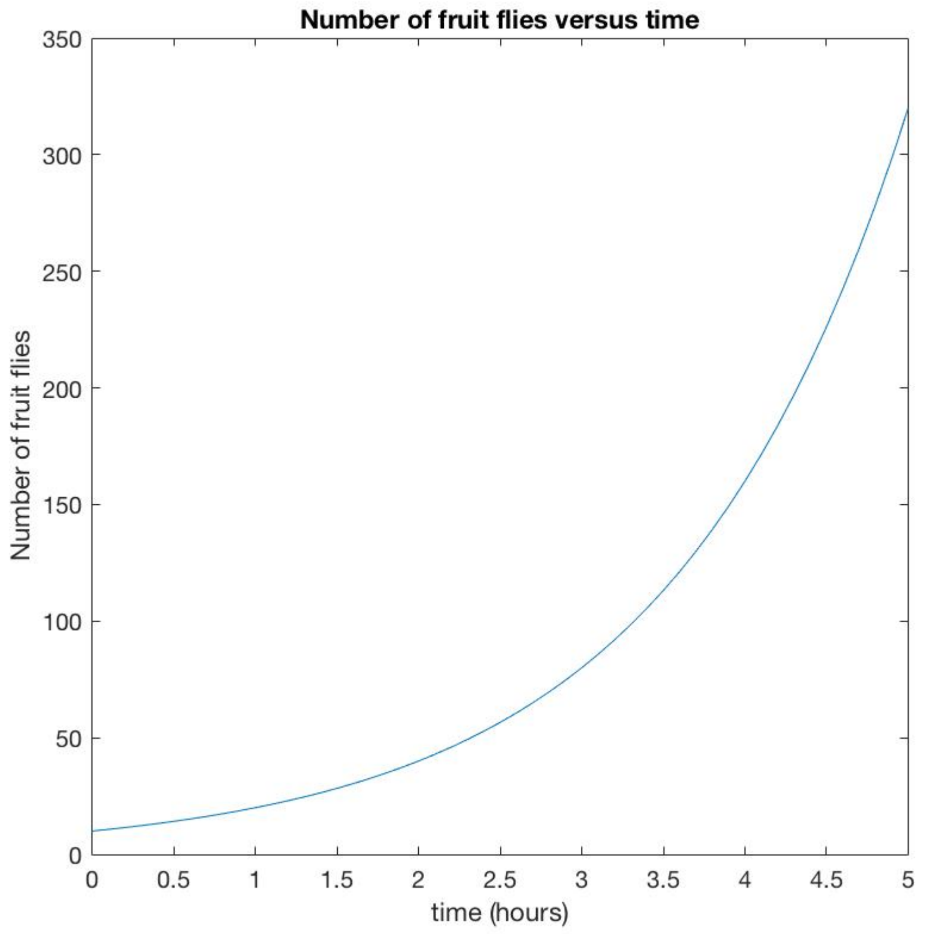


3.2.2 Exponential Modeling

Exponential functions are more complicated than linear functions, but are very useful for things that, for example double in magnitude at a certain rate.

For example, suppose a colony of fruit flies starts with 10, and doubles every hour. Then the population of fruit flies at time t in hours is given as

$$P(t) = 10 \cdot 2^t$$



3.3.0 Domain and Range of a Function

Let $f(x)$ be a function.

- **The *domain* of $f(x)$ is the set of allowable inputs.**
- **The *range* of $f(x)$ is the set of possible outputs for the function.**
- **These can depend on the relationship the functions are modeling, or be intrinsic to the mathematical function itself.**
- **They can also be inferred from the plot of $f(x)$, if it is available.**

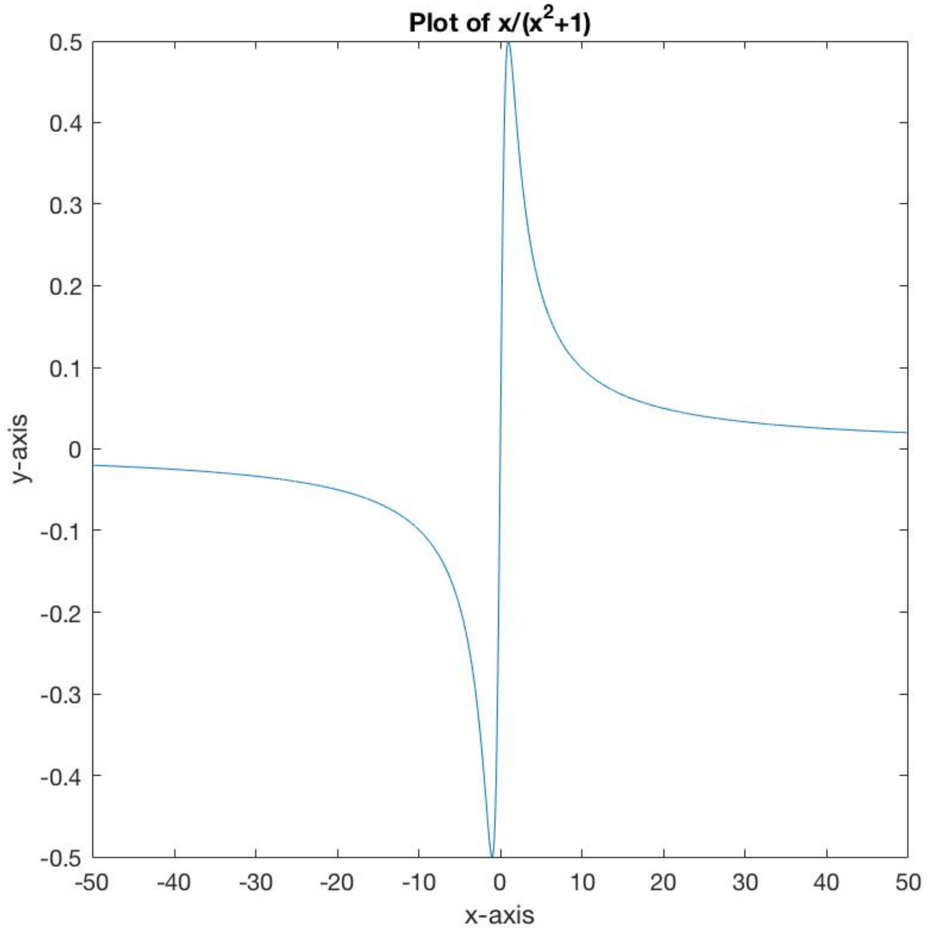
3.3.1 Intrinsic Domain Limitations

Some mathematical objects have intrinsic limitations on their domains and ranges. Classic examples include:

- $f(x) = x^2$ has domain $(-\infty, \infty)$, range $[0, \infty)$.
- $f(x) = \sqrt{x}$ has domain $[0, \infty)$, range $[0, \infty)$.
- $f(x) = \log(x)$ has domain $(0, \infty)$, range $(-\infty, \infty)$.
- $f(x) = a^x$ has domain $(-\infty, \infty)$, range $(0, \infty)$.
- $f(x) = \dfrac{1}{x}$ has domain and range $(\infty, 0) \cup (0, \infty)$.

3.3.2 Visualizing Domain and Range

Given a plot of $f(x)$, one can observe the domain and range by considering what x and y values are achieved.



The function

$$f(x) = \frac{x}{x^2 + 1}$$

is hard to analyze, but its plot helps us guess its domain and range.

3.4.0 Algebra of Functions

- **Functions may be treated as algebraic objects: they may added, subtracted, multiplied, and divided in natural ways.**
- **One must take care in dividing by functions that can be 0. Division by 0 is not defined.**
- **There is one important operation of functions that does not apply to numbers: the operation of *composition.***
- **In essence, composing functions means applying one function, then the other.**

3.4.1 Composition of Functions

Given two functions $f(x), g(x)$, the *composition of* $f(x)$ *with* $g(x)$ is denoted $(f \circ g)(x)$, and is defined as:

$$(f \circ g)(x) = f(g(x)) \text{ .}$$

Similarly, $(g \circ f)(x) = g(f(x))$.

One thinks of $(f \circ g)(x)$ as first applying the rule $g(x)$, then applying the rule $f(x)$.

As an example, consider $f(x) = x + 1, g(x) = x^2$.
By substituting $g(x)$ into $f(x)$, one sees that

$$(f \circ g)(x) = x^2 + 1$$

Similarly, one can substitute $f(x)$ into $g(x)$ to compute that

$$(g \circ f)(x) = (x + 1)^2 = x^2 + 2x + 1$$

In particular, we see that ***composition is not commutative***, i.e.

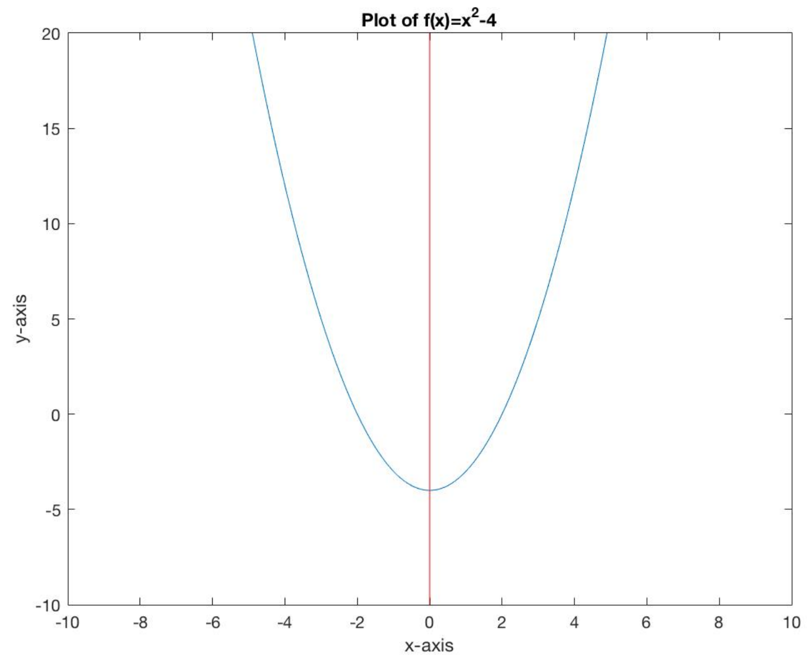
$$(f \circ g)(x) \neq (g \circ f)(x)$$

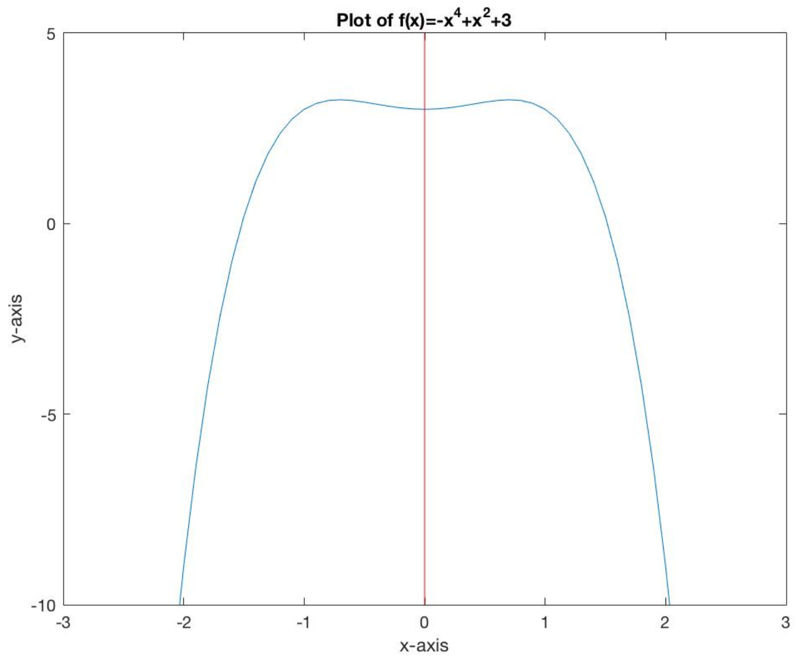
3.5.1 Plotting Functions

- **Drawing a function in the Cartesian plane is extremely useful in understand the relationship it defines.**
- **One can always attempt to plot a function by computing many pairs $(x, f(x))$, and plotting these on the Cartesian plane.**
- **However, simpler qualitative observations may be more efficient. We will discuss of a few of these notions before moving on to some standard function plots to know.**

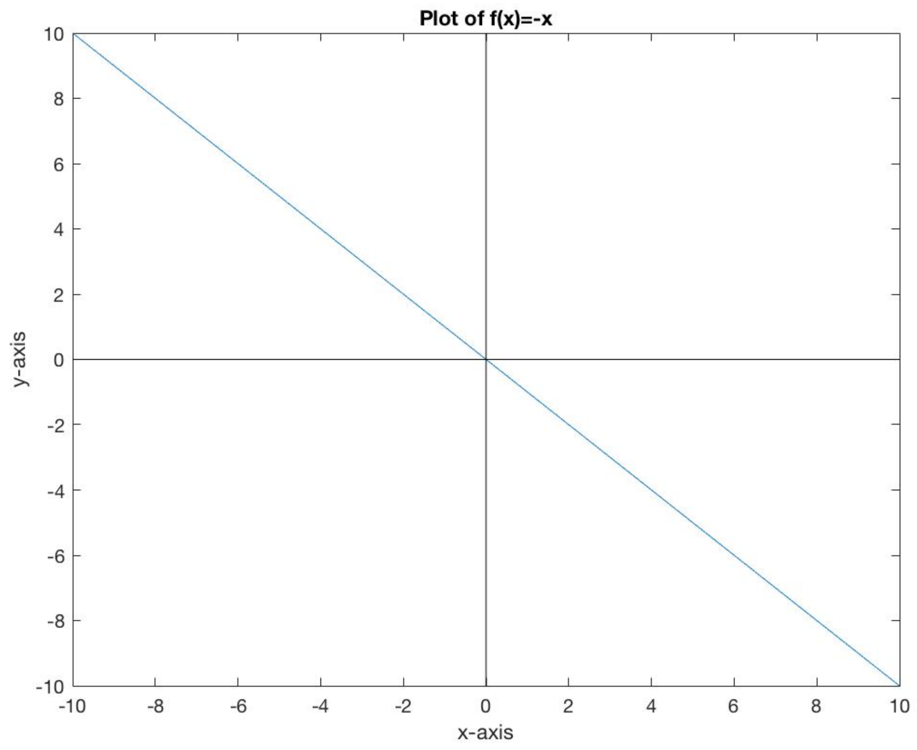
3.5.2 Symmetry of Functions

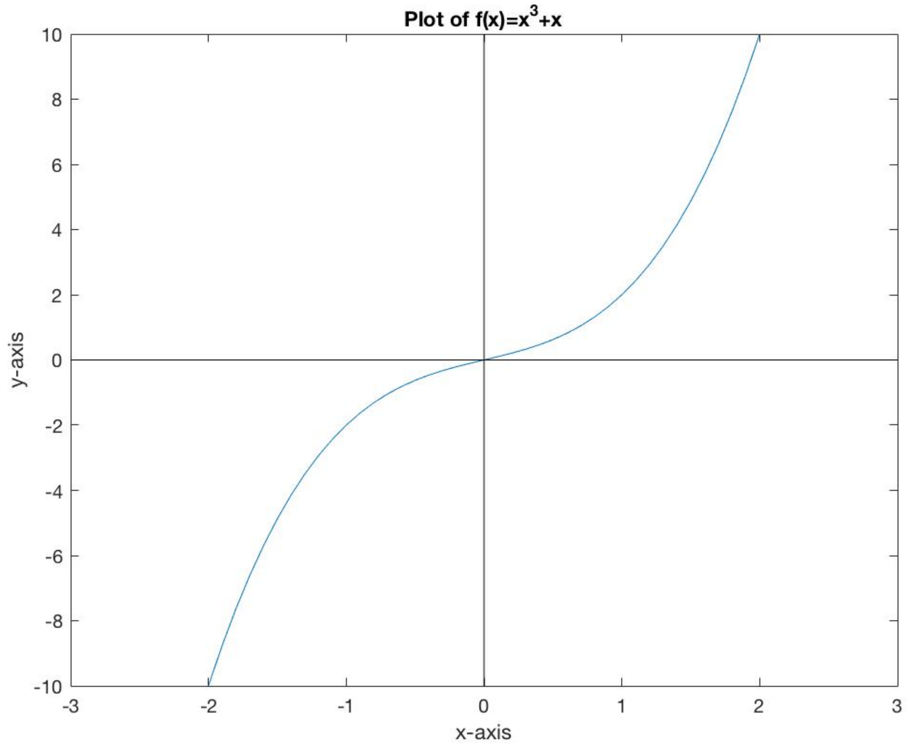
- **A function $f(x)$ is said to be *even/is symmetric about the y-axis* if for all values of $x, f(x) = f(-x)$.**
- **Functions that are even are mirror images of themselves across the y-axis.**





- A function $f(x)$ is said to be ***odd/has symmetry about the origin*** if for all values of x, $f(-x) = -f(x)$
- Functions that are odd can be reflected over the x-axis, then the y-axis.





3.5.3 Transformation of Functions

It is also convenient to consider some standard transformations for functions, and how they manifest visually:

- $f(x) \mapsto f(x+a)$ shifts the function to the left by a if a is positive, and to the right by a if a is negative.
- $f(x) \mapsto f(x)+b$ shifts the function up by b if b is positive, and down by b if b is negative.
- $f(x) \mapsto f(-x)$ reflects the function over the y -axis.
- $f(x) \mapsto -f(x)$ reflects the function over the x -axis.

3.6.0 Inverse Functions

Let $f(x)$ be a function. The *inverse function f* is the function that "undoes" $f(x)$; it is denoted $f^{-1}(x)$.

More precisely, for all x in the domain of $f(x)$,

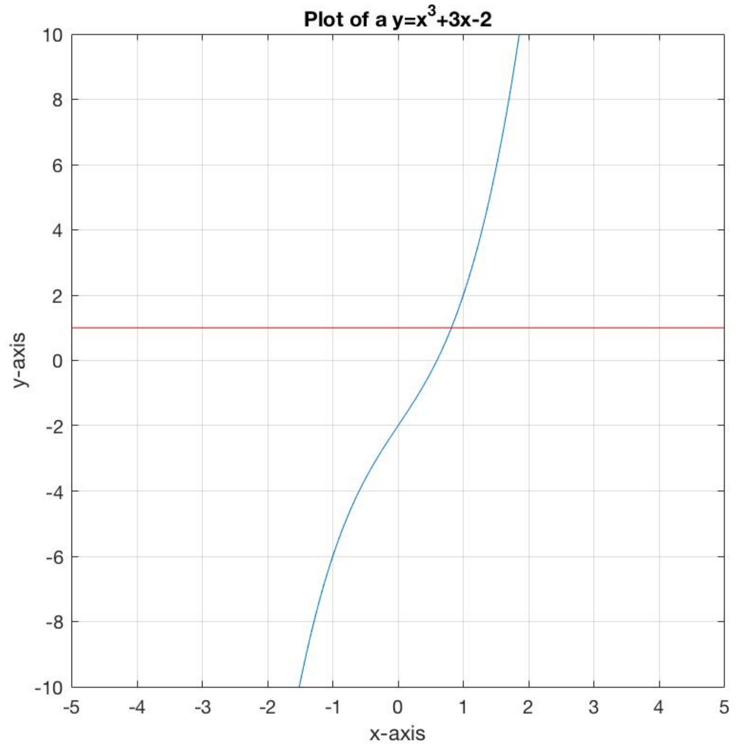
$$(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$$

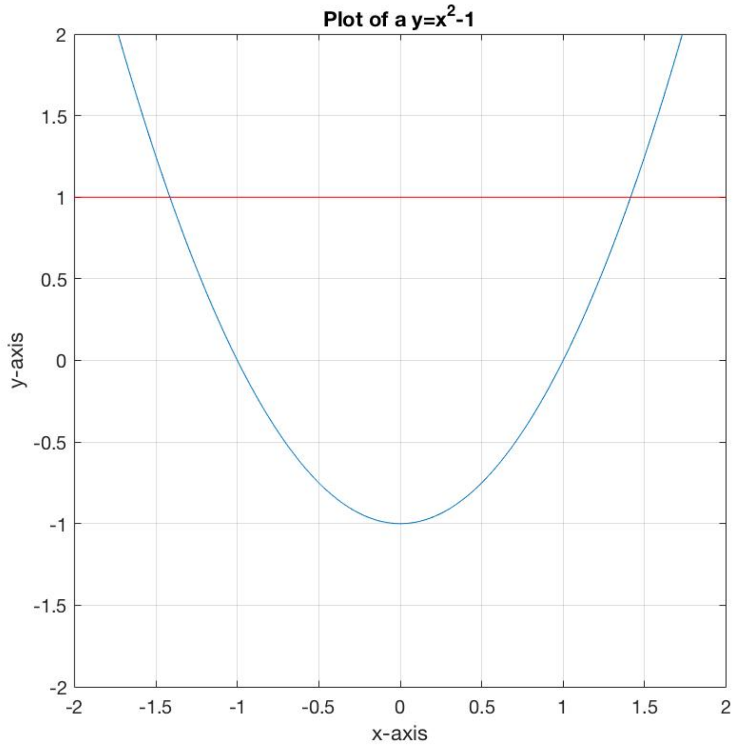
3.6.1 Remarks on Inverse Functions

- **Not all functions have inverse functions; we will show how to check this shortly.**
- **Note that** $f^{-1}(x) \neq (f(x))^{-1}$**, that is, inverse functions are not the same as the reciprocal of a function. The notation is subtle.**
- **The domain of** $f(x)$ **is the range of** $f^{-1}(x)$**, and the range of** $f(x)$ **is the domain of** $f^{-1}(x)$ **.**

3.6.2 Horizontal Line Test

- Recall that one can check if a plot in the Cartesian plane is the plot of a function via the *vertical line test.*
- One can check whether a function $f(x)$ has an inverse function via the *horizontal line test*: the function has an inverse if every horizontal line intersects the plot of $f(x)$ at most once.





4.1.1 Real Numbers

- For us, real numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers.
- There are many subsets of real numbers that are familiar to us.
- Most quantities used to describe things in the world may be understood as real numbers.

$1, 2, 3, \ldots$

$1/5, -1/5, 3/7, \ldots$

$\pi, \sqrt{2}, \log_2(5), \ldots$

4.1.2 Integers

ex: -7 ✓

$7/2$ ✗

$-14/2$ ✓

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

$\mathbb{Z}$

= Zahl → number

4.1.3 Rational Numbers

- Rational numbers are *ratios/fractions* of integers.
- Any number of the form $\frac{p}{q}$ for p, q, integers is rational.
- In particular, every integer is also considered a rational number.
- One must take care: $q = 0$ is not permitted, as this involves division by 0.
- ☆ Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.

ex: $1/2$ ✓

$-81/17$ ✓

$2/1 = 2$ ✓

Remark: Rationals have many representations: $\frac{1}{2} = \frac{2}{4}$

☆ $\frac{p}{0}$ is not defined!

4.1.4 Irrational Numbers

- **There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q.**
- **Such numbers are called *irrational*; there are many of them.**
- **Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$**
- **These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat.***

ex: Classify the following:

- 9π
- $3/77$
- $\pi + 2$
- $\sqrt{2}$
- 17
- -3
- $\sqrt[3]{8}$
- $1 + \frac{1}{5}$
- $\frac{2}{3} + \frac{4}{3}$
- π^0

- $\pi^2 + \frac{1}{\pi^2}$
- $\sqrt{49}$
- $\left(\frac{1}{2}\right)^{100}$
- $(-1)^{100}$

4.2.1 Complex Numbers

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4.2.1 Complex Numbers

ex:
- $1+i$
- $2-7i$
- $3i$
- 4

$i^2+1=0$ i.e. $x^2+1=0$ has $\pm i$ as its roots

- Complex numbers extend the real numbers by introducing the imaginary unit $i=\sqrt{-1}$.
- A complex number is of the form $a+bi$, where a, b are real numbers.
- Complex numbers appear naturally in the context of roots of quadratic polynomials. Recall that

$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\boxed{\text{QF}}$

- If $\underbrace{b^2-4ac}_{\text{discriminant}}<0$, then the roots are complex.

ex: Plot each of the following in the complex plane

like Cartesian plane

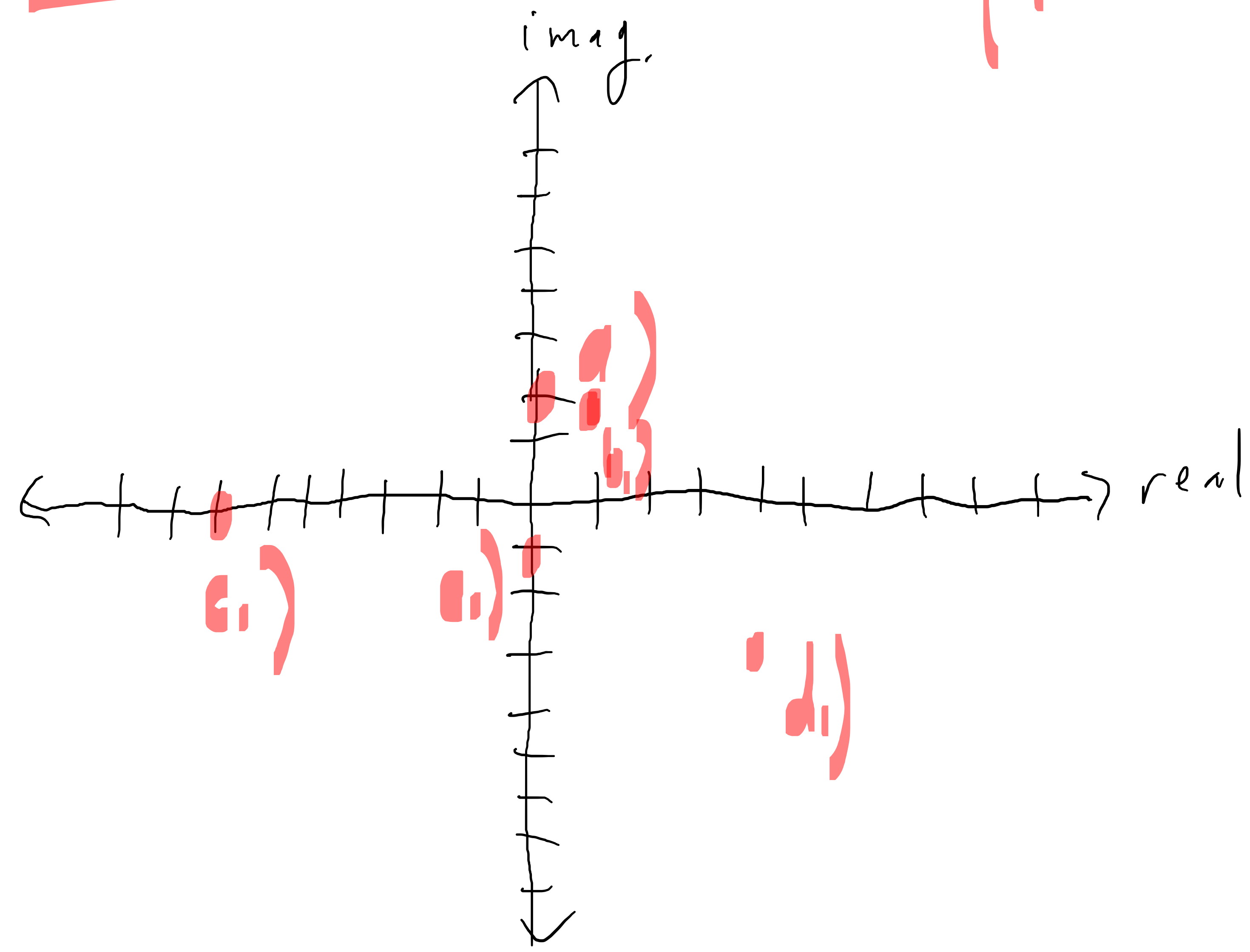
a.) $2i$

b.) $2i+1$

c.) -7

d.) $4-3i$

e.) $-i$



x-axis

↓

real axis

y-axis

↓

imaginary axis

4.2.2 Arithmetic with Complex Numbers

- **Adding and subtracting complex numbers is straightforward:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- **Multiplying requires using the fact that** $i^2 = -1$ **:**

$$\begin{aligned}(a + bi)(c + di) =& ac + adi + bci + bdi^2 \\ =& (ac - bd) + (ad + bc)i\end{aligned}$$

: Simplify each of the following:

$(3+i)(-2-i)$

b.) $(4+2i)^2$

) $(1+i)^2(1-i)$

d.) $i(i+1)$

4.2.3 Division of Complex Numbers

ex: $\overline{1-2i}$

$= 1+2i$

- **Dividing by complex numbers is also tricky. It is convenient to introduce the *conjugate* of a complex number:**

$$\overline{a+bi} = a - bi$$

switch sign of imaginary part

- **With this, we may write**

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}\end{aligned}$$

remove imaginary part

multiply top and bottom by $\overline{c+di} = c - di$

ex: Simplify, by removing the imaginary part:

a.) $\dfrac{1}{i+1} = \dfrac{-i+1}{(i+1)(-i+1)} = \dfrac{-i+1}{-i^2+1} = \dfrac{-i+1}{2}$

b.) $\dfrac{2i-1}{3i+2} = \dfrac{(2i-1)(-3i+2)}{(3i+2)(-3i+2)} = \dfrac{-6i^2+6i+3i-2}{-9i^2+4} = \dfrac{9i+4}{13}$

c.) $\dfrac{2}{2i+4} = \dfrac{2(-2i+4)}{(2i+4)(-2i+4)} = \dfrac{-4i+8}{-4i^2+16} = \dfrac{-4i+8}{20} = \dfrac{-i+2}{5}$

4.3.1 Sequences and Series

4.3.1 Sequences and Series

- *Sequences* are lists of numbers in a given order.
- *Series* are sums of sequences.
- Both sequences and series may be finite or infinite.
- Sequences and series have applications to all fields of science and engineering, and are invaluable tools in finance and business.
- We will look at some special examples of these objects, as opposed to stating a general theory.
- Calculus is the home of the general theory of sequences and series.

$$\left.\begin{array}{l} a_1 = 1 \\ a_2 = 3/2 \\ a_3 = 2 \\ a_4 = 5/2 \\ \vdots \end{array}\right\}$$

4.3.2 Notation

- A sequence of numbers is usually written of the form
$$a_1, a_2, a_3, ..., a_n = \{a_k\}_{k=1}^{n}$$
1 2 3

$k=1 \rightarrow a_1$
$k=2 \rightarrow a_2$
$\vdots$
$k=n \rightarrow a_n$

- This indicates the order, via the subscript, as well as the number of elements in the sequence.
- A series is denoted

add
$$a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$$

- The Greek letter $\sum$ is a capital sigma, and it tells us we will sum everything up.

4.3.3 Arithmetic Series

ex: 1, 3, 5, 7, 9, 11, ... $a_k = 2k-1$

ex: 3, 6, 9, 12, 15, ... $a_k = 3k$

- Many special types of series exist.
- One simple one is *arithmetic series* of the form

$$\sum_{k=1}^{n} ak + b$$

a, b fixed/constant

k changes

- Here, a, b are fixed. This series has the nice property that the differences between terms are fixed at a .
- A classical mathematical exercise in proof-based mathematics is to show that if $\{a_k\}_{k=1}^{n}$ is an arithmetic sequence as above, then

average first and last element

$$\sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}$$

ex: $\sum_{k=1}^{100} k = 100 \cdot \frac{(1+100)}{2}$

Gauss

$$= \frac{100 \cdot 101}{2}$$

$$= 50 \cdot 101$$

$$= 5050 \checkmark$$

ex: $\sum_{j=1}^{50} (2j-1) = 1+3+5+7+\dots$

$$= 50 \cdot \left(\frac{1+99}{2}\right)$$

$$= 50 \cdot \frac{100}{2}$$

$$= 50 \cdot 50$$

$$= 2500 \checkmark$$

ex: $\sum_{j=1}^{N} j = N \cdot \frac{1+N}{2}$

N is variable $= \frac{N(N+1)}{2}$

4.3.4 Geometric Series

ex: $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$= 1$

- Another important special series is the *geometric series* $\sum_{k=1}^{n} r^k$ for some $r < 1$.

- There are nice formulas for both the finite and infinite versions of the geometric series:

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r} \qquad \sum_{k=1}^{\infty} \frac{r}{1-r}$$

$\underline{ex}$: $\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$ $\frac{1}{2} < 1 \Rightarrow$ geo. Series apply

$$= \frac{\frac{1}{2}\left(1-\frac{1}{2}^{10}\right)}{1-\frac{1}{2}}$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \left(1 - \frac{1}{1024}\right)}{\cancel{\frac{1}{2}}} = \frac{1023}{1024} \approx 1$$

ex: $\sum_{j=1}^{\infty} \left(\frac{1}{3}\right)^j = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots \frac{1}{2}$

$$\| \quad \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{2/3} = \frac{1}{2} \checkmark$$

$\underline{ex}$: $\sum_{j=1}^{\infty} 2 \cdot \left(\frac{7}{8}\right)^j$

$\frac{7}{8} < 1$

$$= 2 \sum_{j=1}^{\infty} \left(\frac{7}{8}\right)^j$$

$$= 2 \cdot \frac{7/8}{1 - 7/8} = 2 \cdot \frac{7/8}{1/8}$$

$$= 2 \cdot 7 = 14 \checkmark$$

4.4.1 Factorials and Binomial Theorem

- **Counting problems** are among the most important, and challenging, problems in mathematics.
- When discussing the number of combinations or groups possible from some larger set, the notions of **factorial** and **binomial coefficient** play a crucial role.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

4.4.2 Factorial!

- **The factorial of a number is simply the product of itself with all positive integers less than it.**
- **By convention,** $0! = 1$ **.**
- **It is possible to define the factorial for non-integers, but this quite advanced and is not part of the CLEP.**
- **When computing with factorials, it is helpful to *write out the multiplication explicitly*, as there are often cancellations to be made.**

ex: Compute

a.) $\binom{5}{3}$

b.) $2!$

c.) $\binom{n}{n-1}$

d.) $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$

e.) $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

4.4.3 Counting with Factorials

- Factorials are useful for *combinatorics*, i.e. problems involving counting.
- Given n objects, the number of groups of size k when *order doesn't matter* is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

binomial coefficient

- Given n objects, the number of groups of size k when *order matters* is .

$$\frac{n!}{(n-k)!} \geq \binom{n}{k}$$

ex: 10 people are at a dance. How many ways to choose pairs?

$n = 10$

$k = 2$

order doesn't matter $\longrightarrow$ $\binom{10}{2}$

\# pairs:

$$= \frac{10!}{\underbrace{(10-2)!}_{8!}\cdot 2!} = \frac{10\cdot 9\cdot \cancel{8 \cdots 2\cdot 1}}{\cancel{8\cdot 7\cdot 6 \cdots 1}\cdot 2} = \frac{90}{2} = 45$$

ex: a.) How many unique order triples of letters exist if each letter may be used only once?

$n = 26$ (corresponds to # letters)

$k = 3$ order matters: $\frac{26!}{(26-3)!} = \frac{26!}{23!} = \cdots = 26 \cdot 25 \cdot 24$

$= 15600$

b.) What if order doesn't matter?

$n = 26$

$k = 3$ order doesn't matter: $\binom{26}{3} = \frac{26!}{23! \cdot 3!}$

$= 2600$

4.4.4 Binomial Theorem

ex: $(x+1)^2 = x^2+2x+1$

ex: $(x+1)^{10}$ is hard to do without binomial theorem

- **Another important counting application of factorials is in determining the expansion of the binomial:**

$$(x+y)^n$$

$(x+1)^2$ $n=2$ $y=1$

- **The *binomial theorem* states**

binomial coefficient

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

expand?

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k$$

ex: Expand $(x-y)^5$ $\quad 5 = n$

binomial theorem

$$= \sum_{k=0}^{5} x^k (-y)^{n-k} \cdot \binom{n}{k}$$

$$= x^0(-y)^5\binom{5}{0} + x^1(-y)^4\binom{5}{1} + x^2(-y)^3\binom{5}{2} + x^3(-y)^2\binom{5}{3}$$

$$+ x^4(-y)^1\binom{5}{4} + x^5(-y)^0\binom{5}{5}$$

$$= -y^5 + xy^4\cdot 5 - x^2y^3\cdot 10 + x^3y^2\cdot 10 - x^4y\cdot 5 + x^5$$

ex: Find the 13^{th} term of $(a+b)^{27}$

$$(a+b)^{27} = \sum_{k=0}^{27} \binom{27}{k} a^k b^{27-k}$$

$k = 12$

$$\binom{27}{12} a^{12} b^{27-12}$$

$$= \underbrace{\binom{27}{12}}_{\text{some large number that we won't compute}} a^{12} b^{15}$$

4.5.1 Matrices

ex: $\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$ 3×3

ex: $\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} -1 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$ 2×3

- **A matrix is an array of numbers.**
- **A matrix M is $m \times n$ if it has m rows and n columns.**
- **Matrices with only one row or one column are sometimes called *vectors*.**
- **A matrix with one row and one column is just a number!**
- **Note that an $m \times n$ matrix has mn entries.**

rows columns

ex: $(1, 0, 3)$

ex: (1)

ex: State The dimension:

a.) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ 2×2

b.) $(-1, -1, 0)$ 1×3

c.) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2×1

d.) $\begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ -7 & 4 & 6 & 1 \end{pmatrix}$ 3×4

e.) (0) 1×1

f.) $\cancel{1 \times 2}$ $(1, 2)$ 1×2

4.5.2 Matrix Algebra

$$2\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}2\cdot 1 & 2\cdot 0\\ 2\cdot 0 & 2\cdot 1\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

same # rows
same # columns

- Matrices may be added and subtracted ***only if they are the same size.*** In this case, one simply adds or subtracts the corresponding entries of the matrix.
- Matrices may also be multiplied by single numbers; this is called ***scalar multiplication***, and has the impact of multiplying every entry by the scalar.
- Multiplying and dividing matrices is also possible, and is extremely important in modern mathematics and engineering. It is subtle, and not likely to appear on CLEP.

ex: Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ 0 & 1 \end{pmatrix}$$

ex: Compute

$$2\begin{pmatrix}3\\6\\0\end{pmatrix}+3\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}6\\12\\0\end{pmatrix}+\begin{pmatrix}3\\3\\-3\end{pmatrix}$$

$$=\begin{pmatrix}9\\15\\-3\end{pmatrix}$$

4.5.3 Determinant

- The determinant is a number corresponding to a matrix.
- It contains important information related to matrix division and using matrices to solve linear systems of equations.
- It is defined for *square matrices*, i.e. matrices with the same number of rows as columns.
- For a 2×2 matrix, the formula is:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

matrices

determinants for higher number of rows/columns is possible but trickier ex: Compute the following determinants:

a.) $\det\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 1 \cdot 2 - (0)(-1) = 2 - 0 = 2$

b.) $\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = 1 \cdot 2 - (3)(2) = 2 - 6 = -4$

c.) $\det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot 0 - 0 \cdot 2 = 0 - 0 = 0$

d.) $\begin{pmatrix} -3 & 6 \\ 4 & -8 \end{pmatrix} = (-3 \cdot -8) - (4 \cdot 6)$
$= 24 - 24 = 0$