

1.1.1 Algebraic Operations

We need to learn how our basic algebraic operations interact.

When confronted with many operations, we follow *the order of operations*:

Parentheses 1
Exponentials 2
Multiplication 3
Division 4
Addition 5
Subtraction 6

ex: Simplify $x^2(x+3) - (x \cdot 2x + 1)$

$$\begin{aligned} &= x^2(x+3) - (2x^2 + 1) \\ &= x^2 \cdot x + x^2 \cdot 3 - (2x^2 + 1) \\ &= x^3 + 3x^2 - (2x^2 + 1) \\ &= x^3 + \underline{3x^2} - \underline{2x^2} - 1 \\ &= x^3 + x^2 - 1 \quad \checkmark \end{aligned}$$

$$\underline{\text{ex}} : (3 \cdot (7^2 \div 49) + 2) \div 5$$

$$= (3 \cdot (49 \div 49) + 2) \div 5$$

$$= (3 \cdot (1) + 2) \div 5$$

$$= (3 + 2) \div 5$$

$$= 5 \div 5$$

$$= 1 \quad \checkmark$$

1.1.2 Manipulating Fractions

Fractions are essential to mathematics

- **Adding fractions:** $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

- **Multiplying fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

- **Improper fractions:** $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

ex: Simplify $\frac{a^{5/2} b^2}{a^3 b}$

ex: Compute $\frac{1}{3} + \frac{7}{5} + \frac{2}{7}$

1.1.3 Powers and Roots

For any positive whole number n , $x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$

If $y = x^n$, we say x is an n^{th} root of y .

Roots undo powers, and vice versa. We denote n^{th} roots as $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Notice that:

$$\begin{aligned} & \left(x^{\frac{1}{n}}\right)^n \\ &= x^{\frac{1}{n} \cdot n} \\ &= x^1 \\ &= x \end{aligned}$$

ex: Simplify $\frac{(\sqrt[4]{9})^8}{4^2 - \sqrt[3]{8}}$

1.2.1 Factoring and Expanding Polynomials

- **Polynomials are functions that are sums of nonnegative integer powers of the variables.**
- **The highest power is called the degree of the polynomial.**
- **Higher degree polynomials are generally harder to understand.**

$$y = x^2 \quad f(x) = x - 1 \quad f(r) = 4r^2 - 9$$

$$y = x^2 + 2x - 1 \quad f(x) = x^6 - x + 1$$

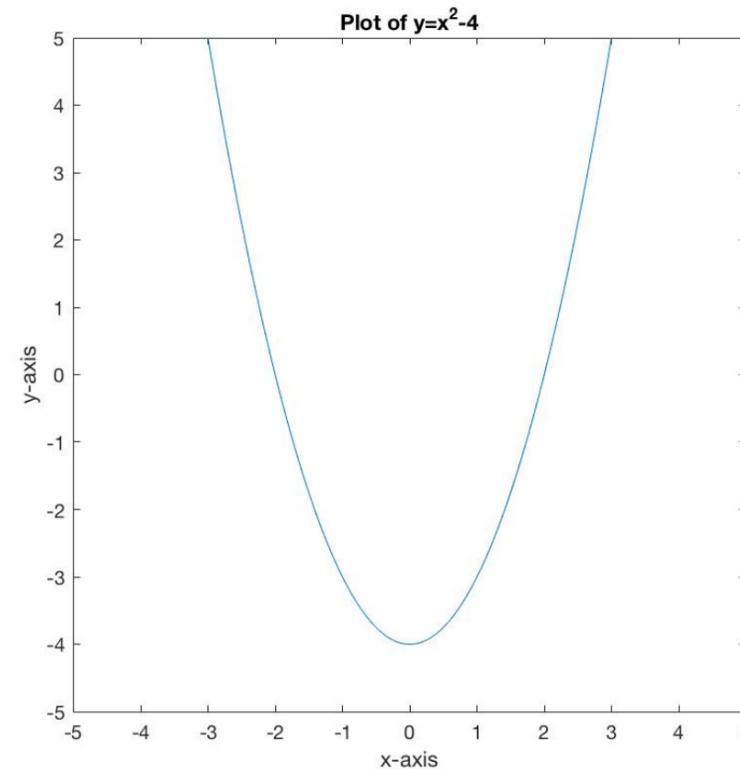
1.2.2 First Order Polynomials

These are just lines:

$$y = ax + b$$

1.2.3 Second Degree Polynomials

$$y = ax^2 + bx + c$$



We can try to **factor** quadratics, i.e. write as a product of first order polynomials

1.2.4 Roots of Quadratics

These can be found by factoring, and also with the famous *quadratic formula*:

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.2.5 Higher Order Polynomials

- **These can also be factored, though it is usually harder.**
- **Formulas like the quadratic formula exist for degree 3,4, polynomials.**
- **Nothing for degree 5 and higher.**

1.2.6 Expanding Polynomials

One can undo factoring by expanding products of polynomials. One must take care with distribution.

$$(ax + b)(cx + d) \neq acx^2 + bd$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

1.3.1 Introduction to Exponentials

$$x^y$$

- The number x is called the base.
- The number y is called the exponent.
- Examples include: x^2 (base x , exponent 2) and 2^x (base 2, exponent x)
- When someone refers to an *exponential function*, they mean the variable is in the exponent (i.e. 2^x), not the base (x^2)

$$e^x = 4^{(x^2)}$$

base = 4

exponent = x^2



1.3.2 Properties of Exponents

Basic Rules:

- (same base, different exponents)

$$a^{x+y} = a^x a^y$$

- (different base, same exponent)

$$a^x b^x = (ab)^x$$

- (iterated exponents)

$$(a^x)^y = a^{xy}$$

- (for any value of x, by convention)

$$x^0 = 1$$

→ already seen
with degree 0 polynomials...

ex: =

$2^2 \cdot 2^2 = 2^4 = 16$

ex: Simplify

$$2^{-3} \cdot 2^3$$

" 2^{-3+3}

" 2^0

" 1

Ex: Simplify

$$2^4 \cdot (2^{-1})^3$$

$$= 2^4 \cdot 2^{-1 \cdot 3}$$

$$= 2^4 \cdot 2^{-3}$$

$$= 2^{4+(-3)}$$

$$= 2^1 = 2 \checkmark$$

ex: Simplify

$$\sqrt{3} \cdot \sqrt{27}$$

$$= 3^{1/2} \cdot 27^{1/2}$$

$$= (3 \cdot 27)^{1/2}$$

$$= (81)^{1/2}$$

$$= \sqrt{81} = 9$$

$$\sqrt{x} = x^{1/2}$$

ex: Simplify

$$a^4 \cdot b^2 \cdot a^{-1/2} \cdot b^{-3}$$

$$= (a^4 \cdot a^{-1/2}) \cdot (b^2 \cdot b^{-3})$$

$$= (a^{4-1/2}) \cdot (b^{2-3})$$

$$= a^{7/2} \cdot b^{-1}$$

$$= a^{7/2} b^{-1}$$

Ex:

Simplify

$$x^{1-y} \cdot x^{y+7} \cdot x^{-6}$$

have base x

$$[(1-y) + (y+7) + (-6)]$$

=

$$x$$

=

$$x^2$$

ex : Simplify

$$3^x \cdot 2^x \cdot 6^{-x}$$

$$= (3 \cdot 2)^x \cdot 6^{-x}$$

$$= 6^x \cdot 6^{-x}$$

$$= 6^{x-x}$$

$$= 6^0 = 1$$

ex: Simplify

$$\frac{x^2 \cdot x^{-3}}{x^5}$$

$$\frac{x^4}{x} \cdot x$$

$$\frac{1}{x} = x^{-1} =$$

$$\frac{x^{2-3}}{x^5}$$

$$\frac{x^4}{x}$$

$$x$$

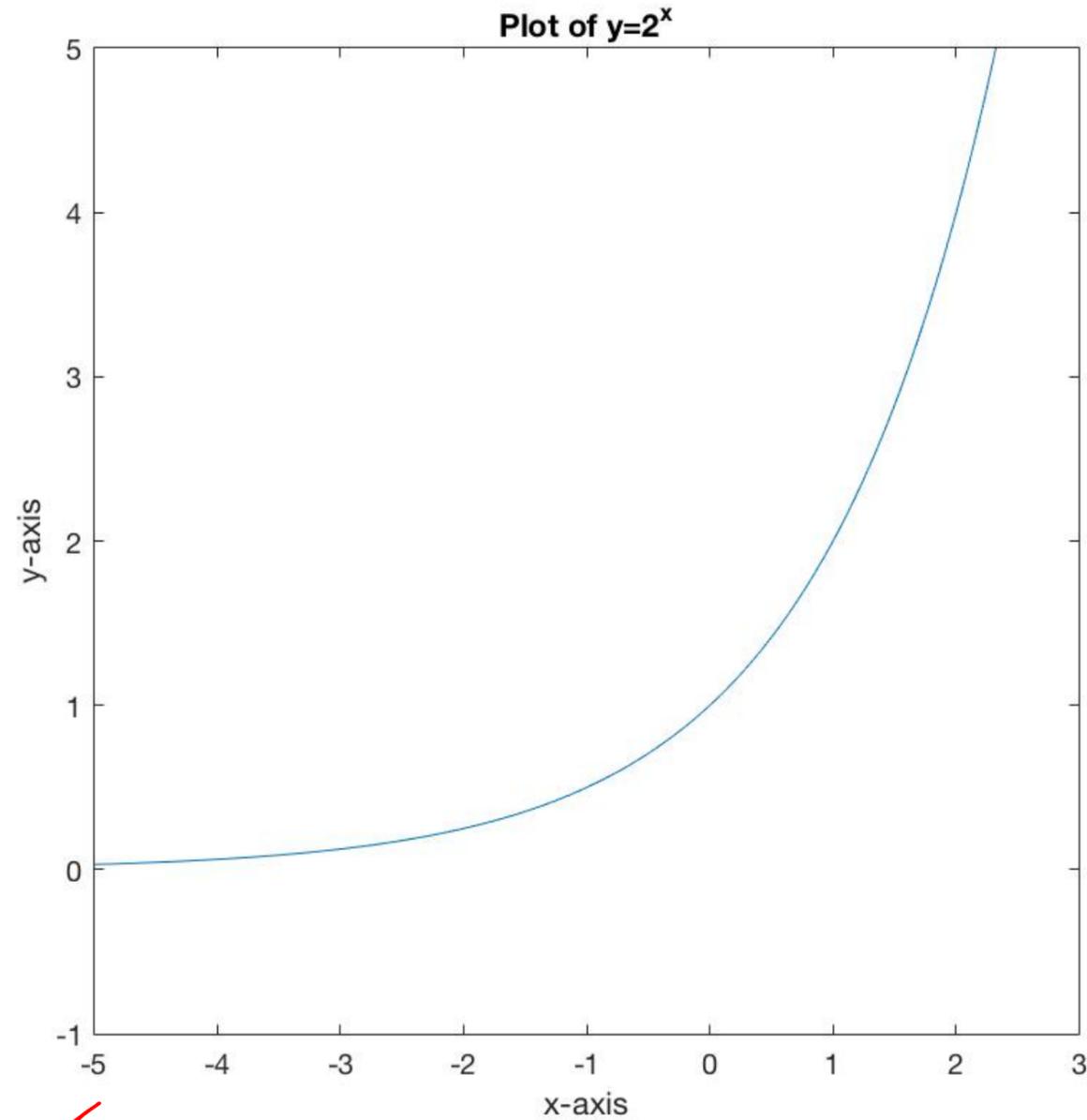
$$= \frac{x^{-1}}{x^5}$$

$$= x^4 =$$

$$\frac{x^4}{x^5} = x^{-1}$$

$$= x^{-1}$$

1.3.3 Plots of Exponentials

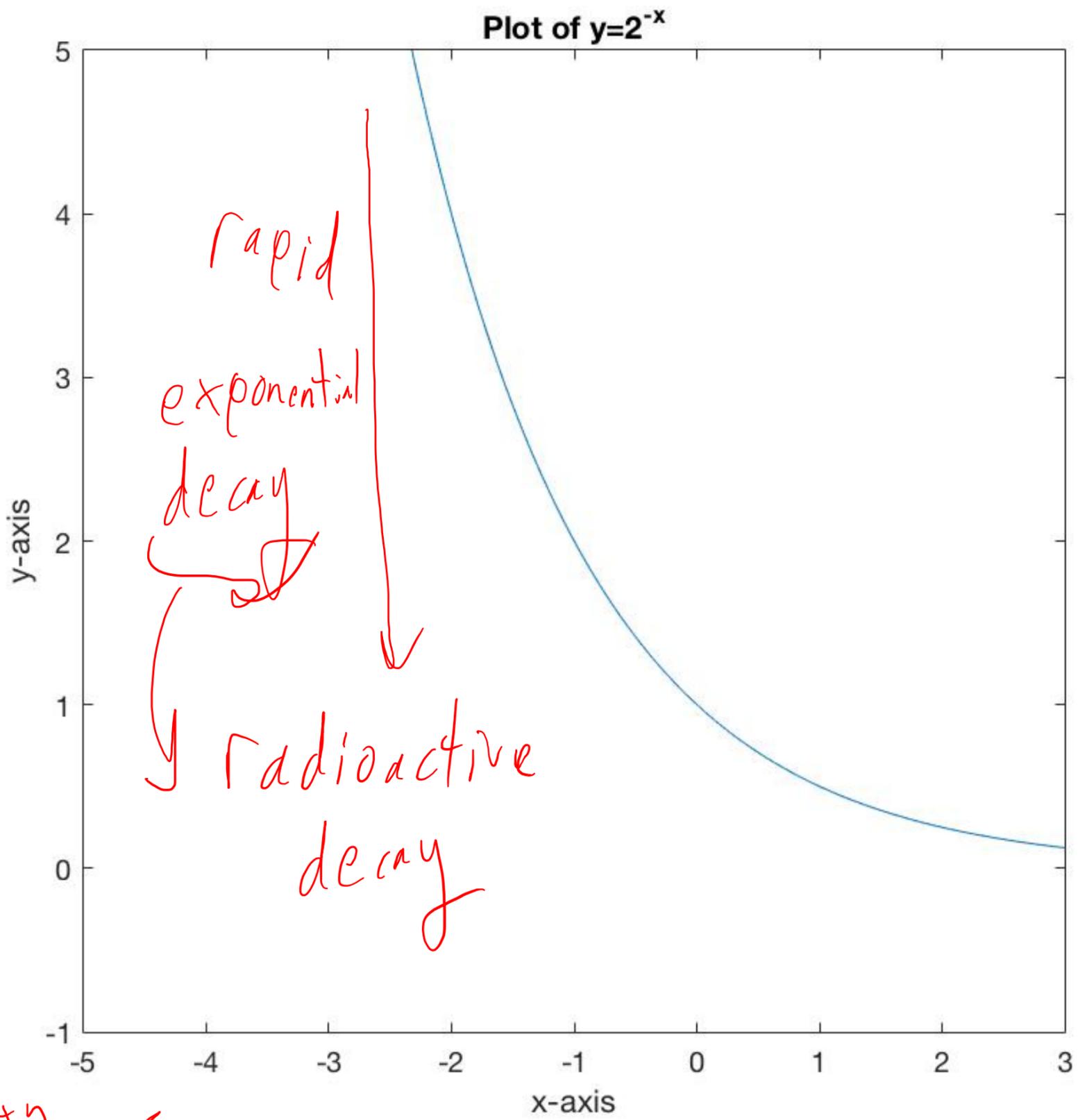


exponential
growth
rapid
growth

x	y = 2 ^x
-1	1/2
0	1
1	2
2	4
3	8



x	$y = 2^{-x}$
-1	$2^{-(-1)} = 2$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



growth



decay

1.4.1 Logarithms

$$y = \log_a(x)$$

- **We call a the base.**
- **Logarithms are a compact way to solve certain exponential equations:**

$$y = \log_a(x) \Leftrightarrow a^y = x$$

1.4.2 Properties of Logarithms

Logarithms enjoy certain algebraic properties, related to the exponential properties we have already studied.

- $\log_a(xy) = \log_a(x) + \log_a(y)$ (logarithm of a product)
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ (logarithm of a quotient)
- $\log_a(x^y) = y \log_a(x)$ (logarithm of an exponential)
- $\log_a(1) = 0$ (logarithm of 1 equals 0)

analogous $a^0 = 1 \iff \log_a(1) = 0$

$$\text{ex: } \log_{10}(x^2 y^3)$$

$$= \log_{10}(x^2) + \log_{10}(y^3)$$

$$= 2 \log_{10}(x) + 3 \log_{10}(y) \quad \checkmark$$

$$\text{Ex: } \log_2(16)$$

$$= \log_a(2^4)$$

$$16 = 2^4$$

$$a' = a$$



$$= 4 \cdot \log_a(2)$$

$\log_a(a) = 1$, for
any choice of a .

$$= 4 \cdot 1 = 4 \quad \checkmark$$

$$\underline{\text{ex}}: \log_4 (2^x \cdot 2^x)$$

$$= \log_4 ((2 \cdot 2)^x)$$

$$= \log_4 (4^x)$$

$$= x \cdot \log_4 (4)$$

$$= x \cdot 1 = x \quad \checkmark$$

$$\underline{ex} : \log_2 \left(\frac{x^2}{8^2} \right)$$

$$= \log_2 (x^2) - \log_2 (8^2)$$

$$= 2 \cdot \log_2 (x) - 2 \log_2 (8)$$

$$= 2 \log_2 (x) - 6 \quad \checkmark$$

ex : $\log_{10} \left(\frac{x^2}{x^4} - x^{-2} + 1 \right)$

$= \log_{10} \left(\cancel{x^{-2}} - \cancel{x^{-2}} + 1 \right)$

$= \log_{10} (1)$
 $= 0$ ✓

Note : $\frac{x^a}{x^4} = x^{-2}$

Fact $\log_a(1) = 0$,
for every base a .

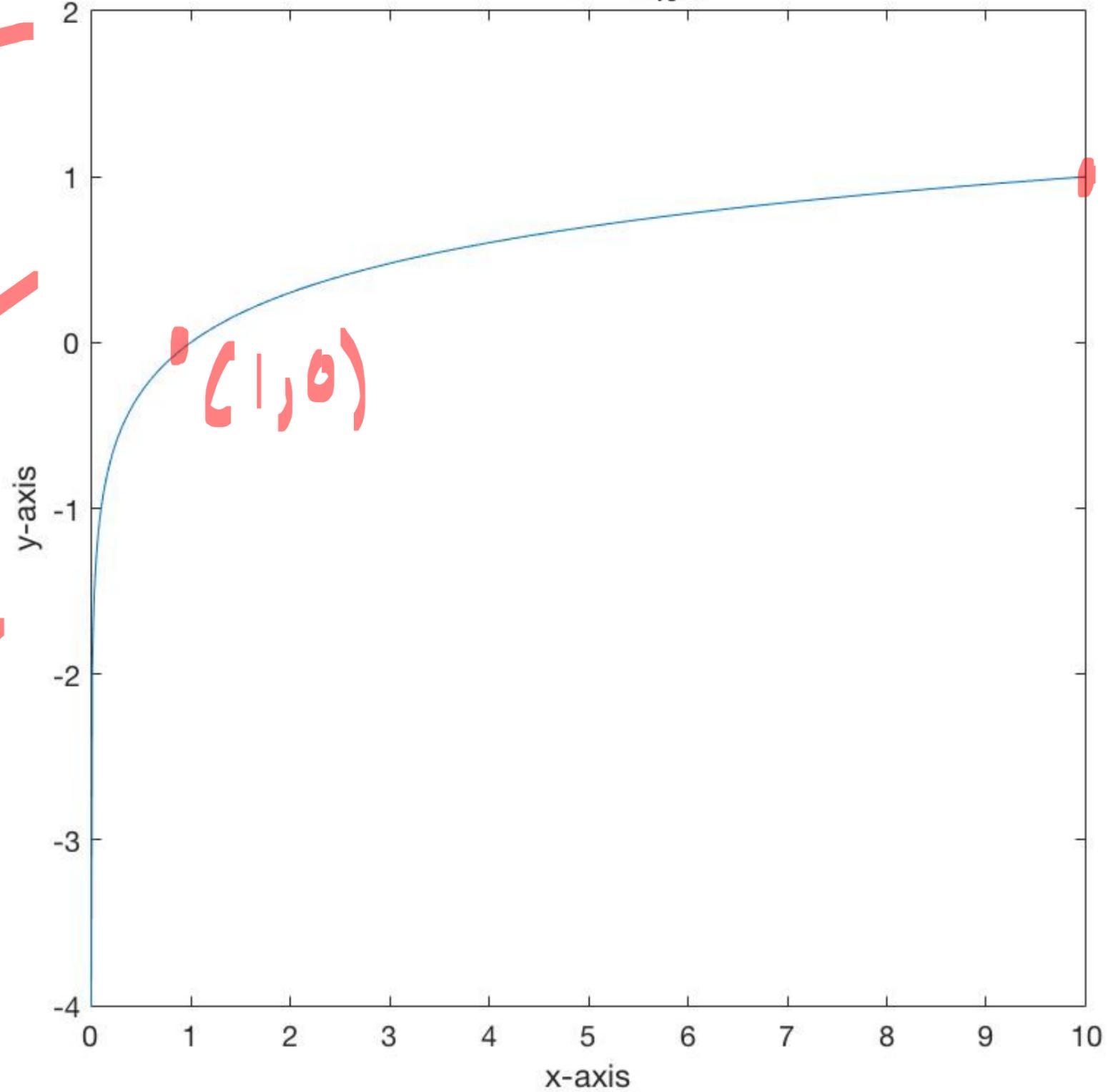
1.4.3 Logarithm as Inverse of Exponential

$$\log_a(a^x) = a^{\log_a(x)} = x$$

i.e., \log_a undoes a^x , vice versa

ex:
 $\log_3(3^4)$
 $= \log_3(3^4)$
 $= 4$

Plot of $y = \log_{10}(x)$



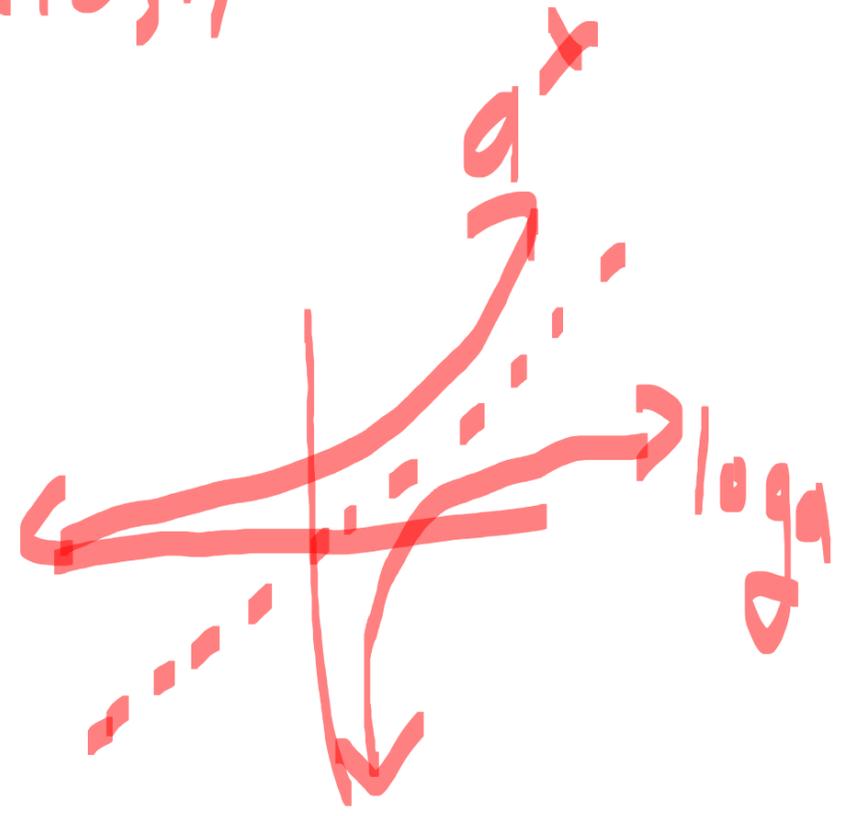
x	y = log ₁₀ (x)
1	0
10	1
100	2

we see
logarithms grow
very slowly

$\log_{10}(7) \approx ?$

(10, 1)

(1, 0)



Sketch

2.1.1 Linear Equations and Inequalities

Equations of the form $y = ax + b$

We want to compute values of x given y and vice versa.

Sometimes we need to perform some algebraic rearrangements first

2.1.2 Linear Inequalities

Linear equations can be broadened to linear ***inequalities*** of the form $y \leq mx + b$, with $\geq, <, >$ potentially in place of \leq

Since $y = mx + b$ defines a line in the Cartesian plane, linear inequalities refer to all points on one side of a line, either including (\leq, \geq) or excluding ($<, >$) the line itself.

2.2.1 Quadratic Equations

Quadratic refers to degree two polynomials. Quadratic equations are equations involving degree two polynomials:

$$y = ax^2 + bx + c$$

Unlike linear equations, in which simple algebraic techniques were sufficient, finding solutions to quadratics requires more sophisticated techniques, such as:

- **Factoring**
- **Quadratic Formula**
- **Completing the Square**

2.2.2 Quadratic Formula

A formulaic approach to solving quadratic equations is the *quadratic formula*:

$$0 = ax^2 + bx + c \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, quadratic equations have two distinct roots, unless $b^2 - 4ac = 0$.

2.2.3 Quadratic Inequalities

$$ax^2 + bx + c \geq 0$$

Solving quadratic inequalities can be made easier with the observation that

$$AB \geq 0 \Leftrightarrow A \geq 0 \text{ and } B \geq 0$$

or $A \leq 0 \text{ and } B \leq 0$

This suggests factoring our quadratic, and examining when each linear factor is positive or negative.

Similarly, $AB \leq 0 \Leftrightarrow A \geq 0 \text{ and } B \leq 0$
or $A \leq 0 \text{ and } B \geq 0$

Again, we see that if we can factor our quadratic into linear factors, we can examine each factor individually.

Indeed, supposing that our quadratic inequality has the form

$$x^2 + bx + c \geq 0,$$

we can factor $x^2 + bx + c = (x - \alpha)(x - \beta)$ and examine the corresponding linear factors.

2.3.1 Exponential and Logarithmic Equations

These may look daunting! However, we can use our exponential and logarithmic properties (tricks) to make our lives easier; see Lecture 1.3,1.4.

Recall that $y = a^x \Leftrightarrow \log_a(y) = x$.

From this, we can approach many equations that look intimidating.

2.4.1 Absolute Value Equations

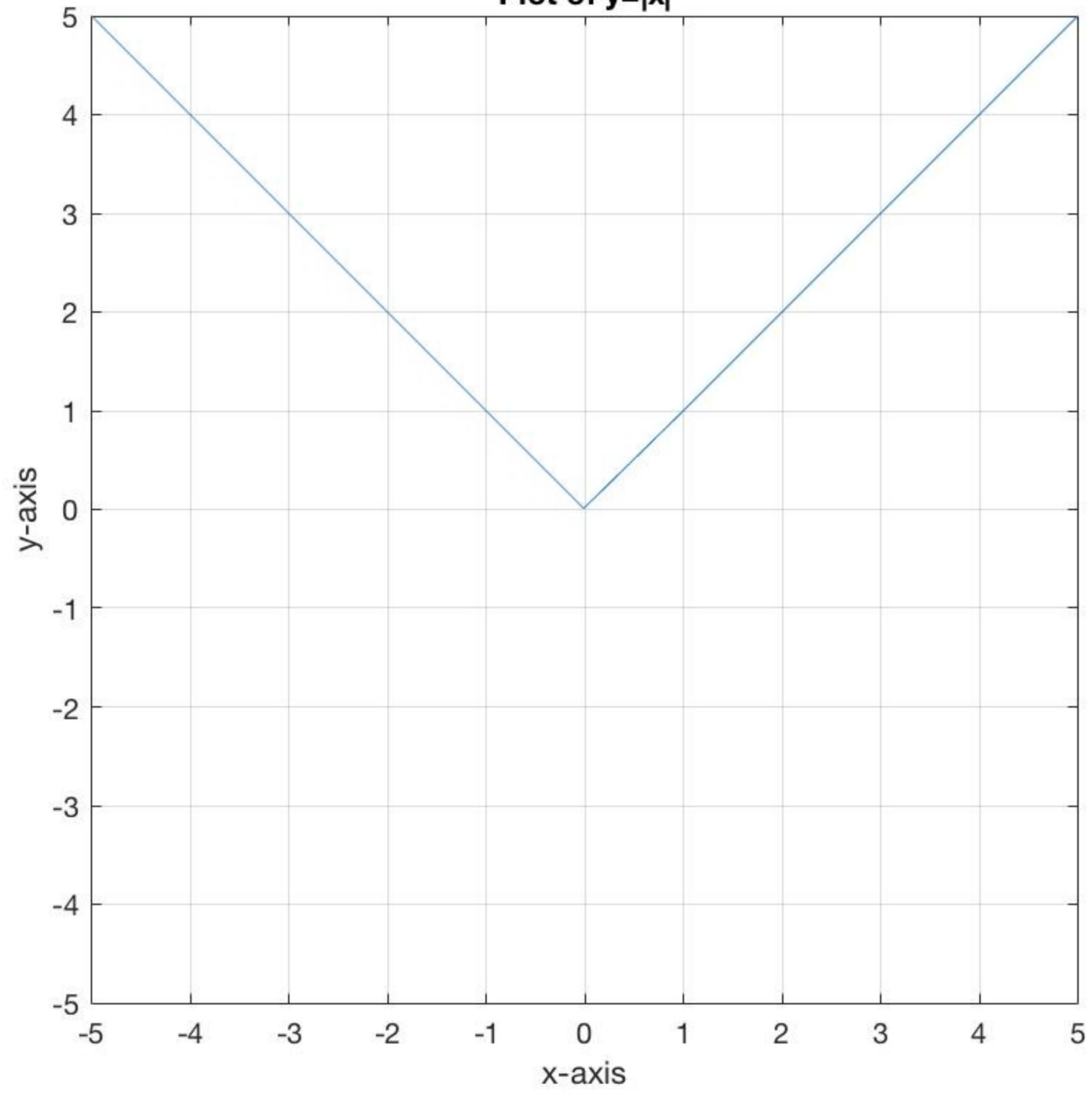
2.4.1 Absolute Value Equations

Recall the absolute value function, which is equal to a number's distance from 0:

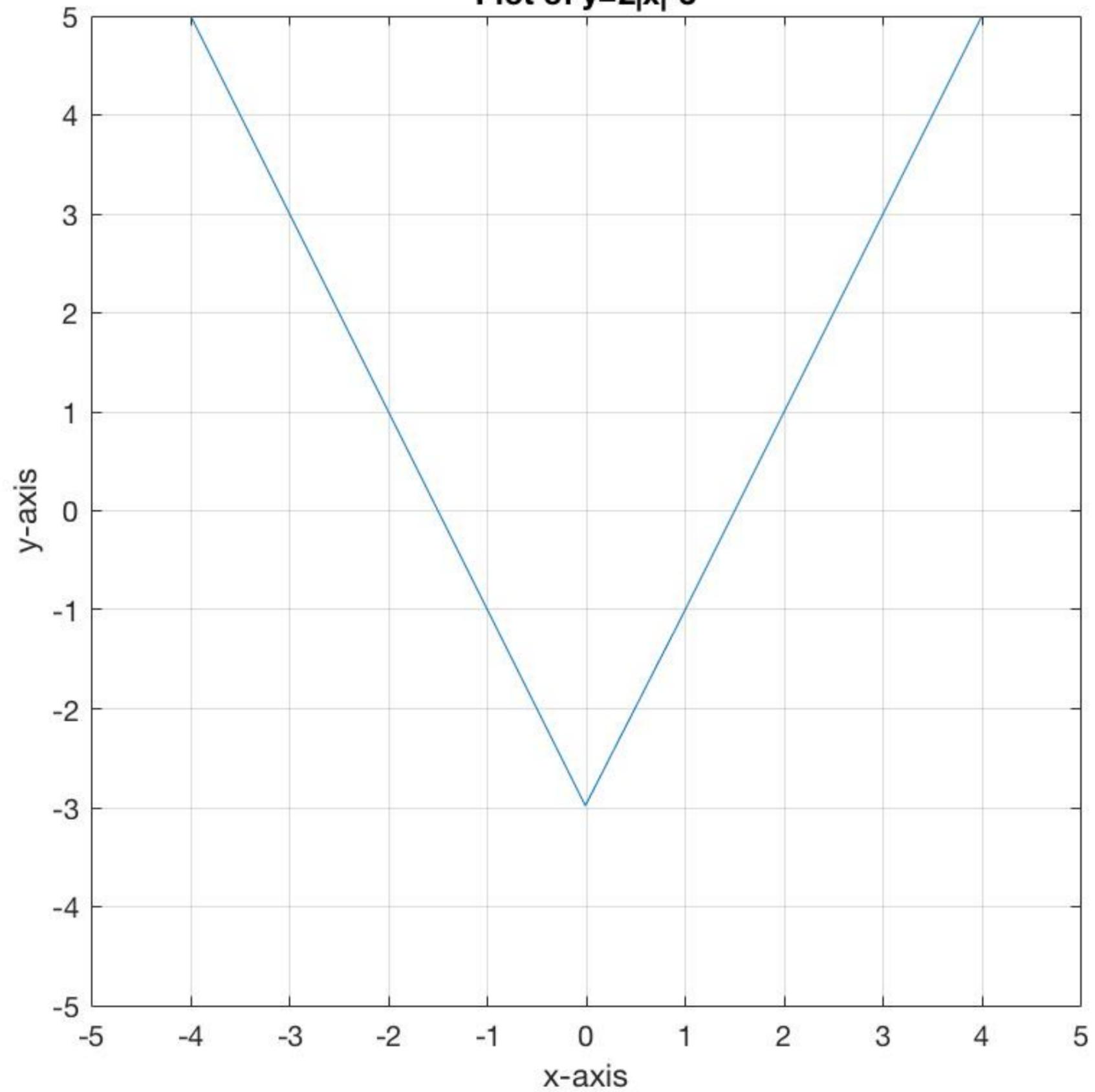
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, the absolute value function keeps positive numbers the same, and switches negative numbers into their positive counterpart.

Plot of $y=|x|$



Plot of $y=2|x|-3$



2.4.2 Equations with Absolute Values

2.4.2 Equations with Absolute Value

When considering equations of the form: $|f(x)| = g(x)$,

it suffices to consider the two cases

$$f(x) = g(x) \text{ and } -f(x) = g(x)$$

In the case of absolute value equations involving first order polynomials (linear functions), we get:

$$|ax + b| = c \Leftrightarrow ax + b = c \text{ or } -(ax + b) = c$$

2.4.3 Inequalities Involving Absolute Value

2.4.4 Linear Absolute Value Inequalities

One can, when working with inequalities of the form

$$|ax + b| \leq c \text{ or } |ax + b| \geq c$$

proceed by finding the two solutions to

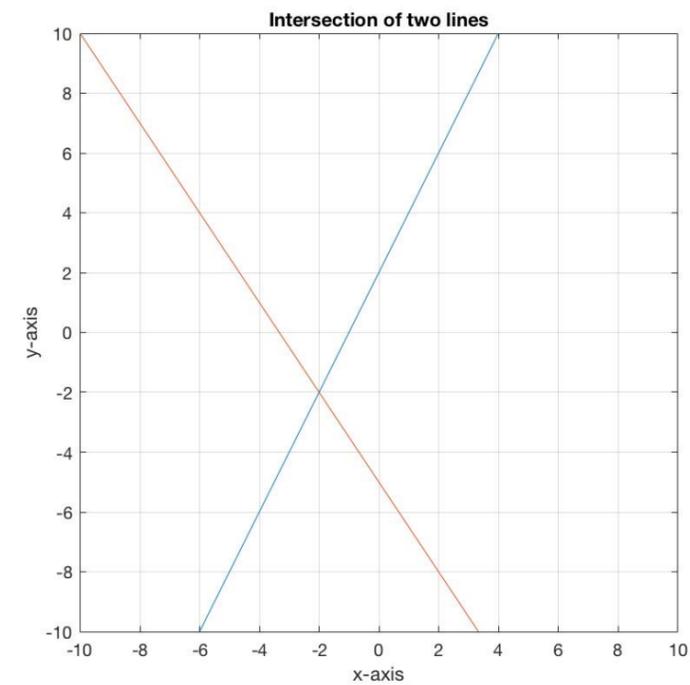
$$ax + b = c \text{ and } -(ax + b) = c$$

then plotting these on a number line, and checking in which region the desired inequality is achieved. This is the *number line method*.

2.5.1 Systems of Equations and Inequalities

A classic area of mathematics is solving two or more systems of equations or inequalities *simultaneously*.

**On classic formulation is:
find the
intersection of
two lines,
given their
equations**



2.5.2 Systems of Linear Equations

The problem of finding the intersection of two lines may be formulated as the algebraic problem of finding the simultaneous solution to a system of linear equations

$$\begin{cases} y = m_1x + b_1 \\ y = m_2x + b_2 \end{cases}$$

Classical solution method: Set the two expressions on the right equal and solve for x , then go back and solve for y .

2.5.3 Higher Order Systems

**It is possible to mix other types of equations into systems.
The same techniques as before work.**

$$\begin{cases} y = 3x + 4 \\ y = x^2 + x + 1 \end{cases}$$

**While more complicated
looking, this system can
be solved with our substitution
method, combined with the
quadratic formula.**

2.5.4 Systems of Inequalities

One can also study regions in the Cartesian plane in which a inequalities are simultaneously satisfied.

In the case of linear inequalities, these may be of the form:

$$\begin{cases} y \leq m_1x + b_1 \\ y \leq m_2x + b_2 \end{cases}$$

$$\begin{cases} y \geq m_1x + b_1 \\ y \leq m_2x + b_2 \end{cases}$$

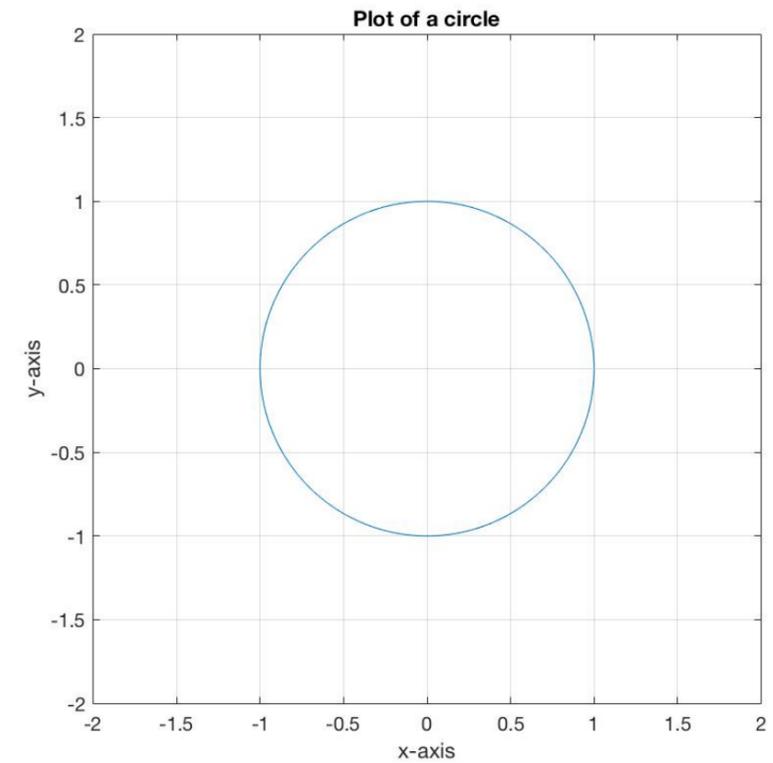
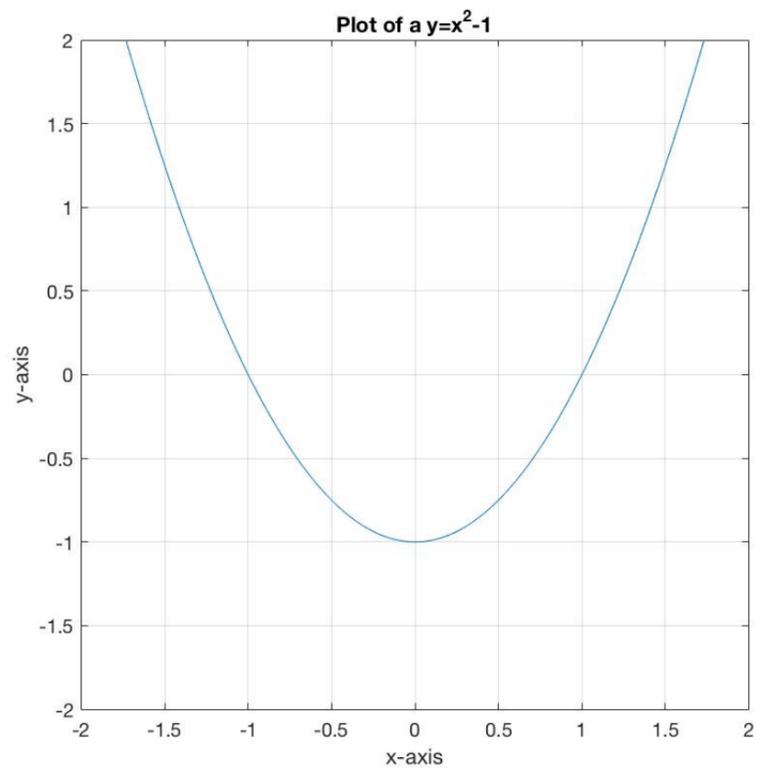
$$\begin{cases} y \geq m_1x + b_1 \\ y \geq m_2x + b_2 \end{cases}$$

3.1.1 What is a Function?

- **Functions are mathematical objects that send an input to a unique output.**
- **They are often, but not always, numerical.**
- **The classic notation is that $f(x)$ denotes the output of a function f at input value x .**
- **Functions are abstractions, but are very convenient for drawing mathematical relationships, and for analyzing these relationships.**

3.1.2 Function or not?

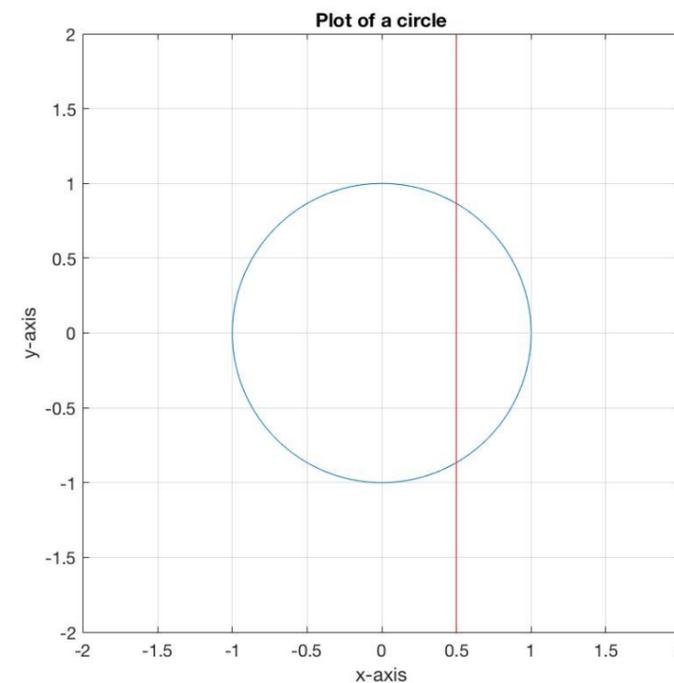
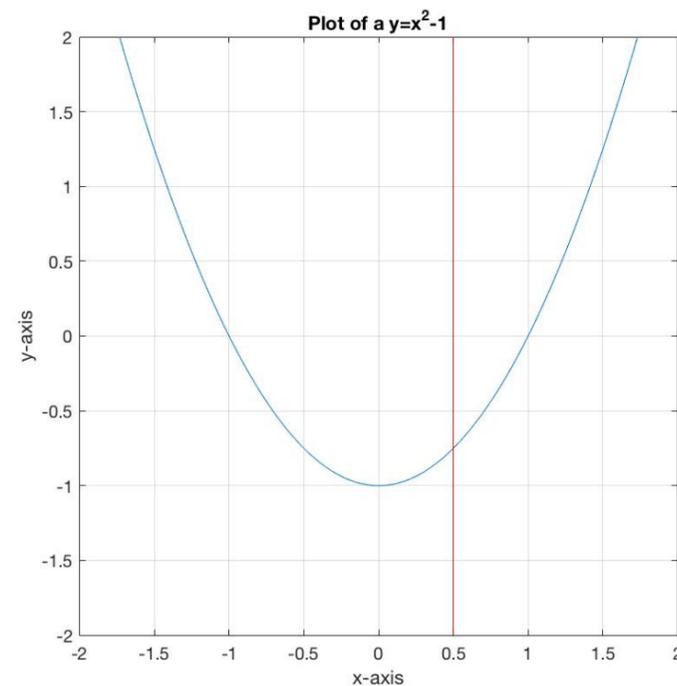
One of the key properties of a function is that it assigns a unique output to an input.



3.1.3 Vertical Line Test

A trick for checking if a mathematical relationship plotted in the Cartesian plane is a function is the *vertical line test*.

VLT: A plot is a function if and only if every vertical line intersects the plot in at most one place.



3.2.0 Representing with Functions

Functions are convenient for describing numerical relationships.

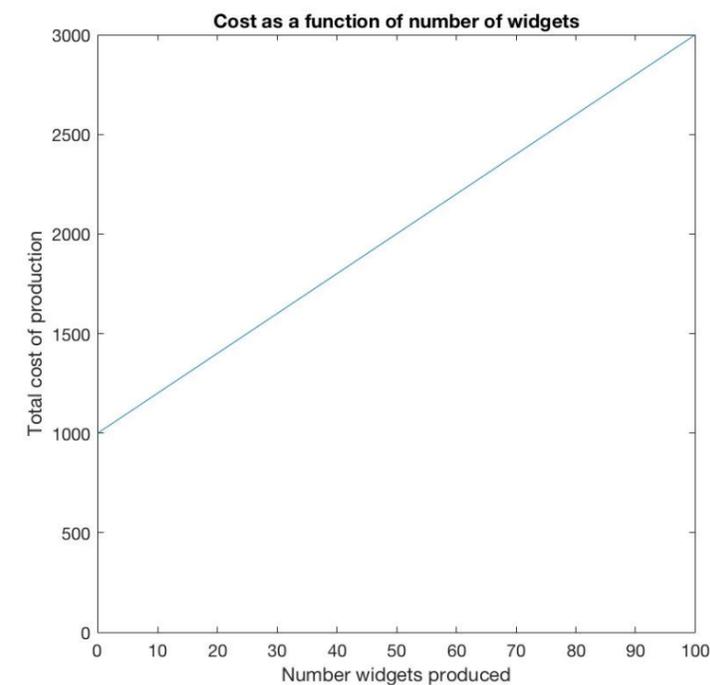
Input \xrightarrow{f} Output

To model a relationship with functions, you simply need to understand how your input depends on your output.

3.2.1 Linear Modeling

Some simple relationships can be modeled with *linear relationships* of the form $f(x) = ax + b$

For example, suppose the cost of producing x widgets is \$1000, plus \$20 for each widget produced. Then the total cost of producing x widgets is modeling as $C(x) = 20x + 1000$

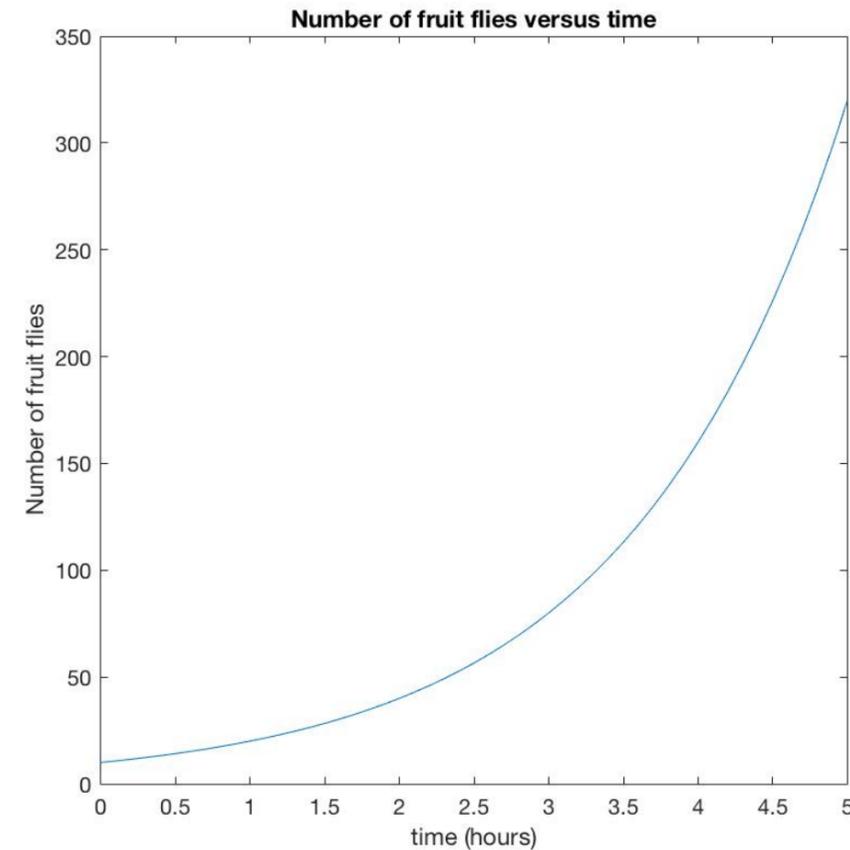


3.2.2 Exponential Modeling

Exponential functions are more complicated than linear functions, but are very useful for things that, for example double in magnitude at a certain rate.

For example, suppose a colony of fruit flies starts with 10, and doubles every hour. Then the population of fruit flies at time t in hours is given as

$$P(t) = 10 \cdot 2^t$$



3.3.0 Domain and Range of a Function

Let $f(x)$ be a function.

- **The *domain* of $f(x)$ is the set of allowable inputs.**
- **The *range* of $f(x)$ is the set of possible outputs for the function.**
- **These can depend on the relationship the functions are modeling, or be intrinsic to the mathematical function itself.**
- **They can also be inferred from the plot of $f(x)$, if it is available.**

3.3.1 Intrinsic Domain Limitations

Some mathematical objects have intrinsic limitations on their domains and ranges. Classic examples include:

- $f(x) = x^2$ has domain $(-\infty, \infty)$, range $[0, \infty)$.
- $f(x) = \sqrt{x}$ has domain $[0, \infty)$, range $[0, \infty)$.
- $f(x) = \log(x)$ has domain $(0, \infty)$, range $(-\infty, \infty)$.
- $f(x) = a^x$ has domain $(-\infty, \infty)$, range $(0, \infty)$.
- $f(x) = \frac{1}{x}$ has domain and range $(\infty, 0) \cup (0, \infty)$.

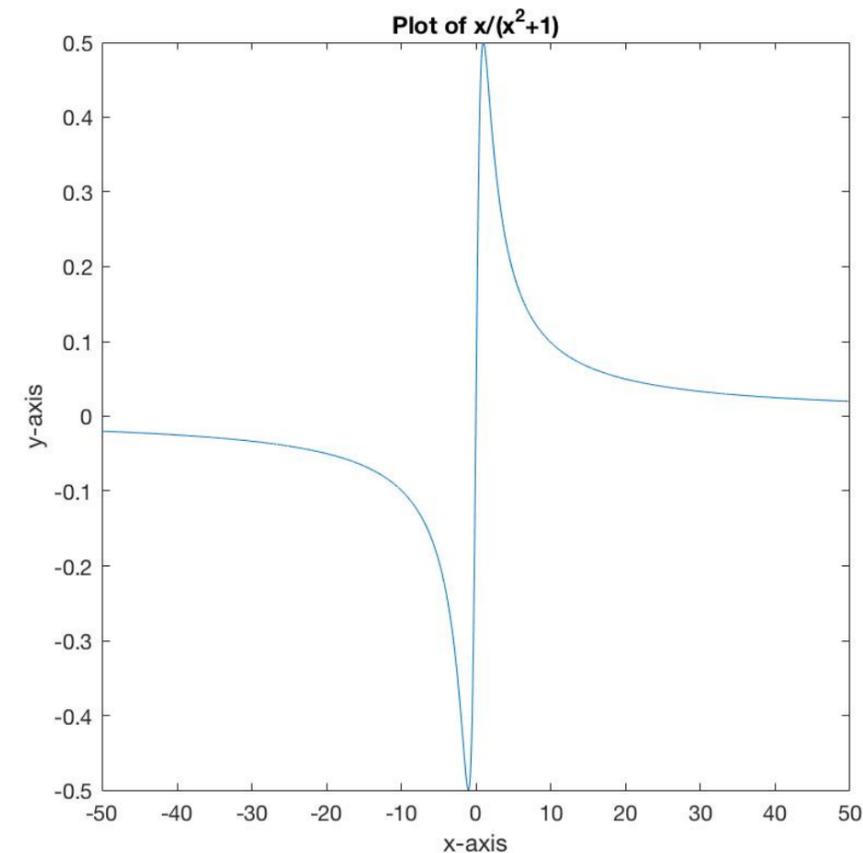
3.3.2 Visualizing Domain and Range

Given a plot of $f(x)$, one can observe the domain and range by considering what values and values are achieved.

The function

$$f(x) = \frac{x}{x^2 + 1}$$

is hard to analyze, but its plot helps us guess its domain and range.



3.4.0 Algebra of Functions

- **Functions may be treated as algebraic objects: they may added, subtracted, multiplied, and divided in natural ways.**
- **One must take care in dividing by functions that can be 0. Division by 0 is not defined.**
- **There is one important operation of functions that does not apply to numbers: the operation of *composition*.**
- **In essence, composing functions means applying one function, then the other.**

3.4.1 Composition of Functions

Given two functions $f(x)$, $g(x)$, the *composition of $f(x)$ with $g(x)$* is denoted $(f \circ g)(x)$, and is defined as:

$$(f \circ g)(x) = f(g(x)) .$$

Similarly, $(g \circ f)(x) = g(f(x))$.

One thinks of $(f \circ g)(x)$ as first applying the rule $g(x)$, then applying the rule $f(x)$.

**As an example, consider $f(x) = x + 1$, $g(x) = x^2$.
By substituting $g(x)$ into $f(x)$, one sees that**

$$(f \circ g)(x) = x^2 + 1$$

**Similarly, one can substitute $f(x)$ into $g(x)$ to
compute that**

$$(g \circ f)(x) = (x + 1)^2 = x^2 + 2x + 1$$

**In particular, we see that *composition is not
commutative*, i.e.**

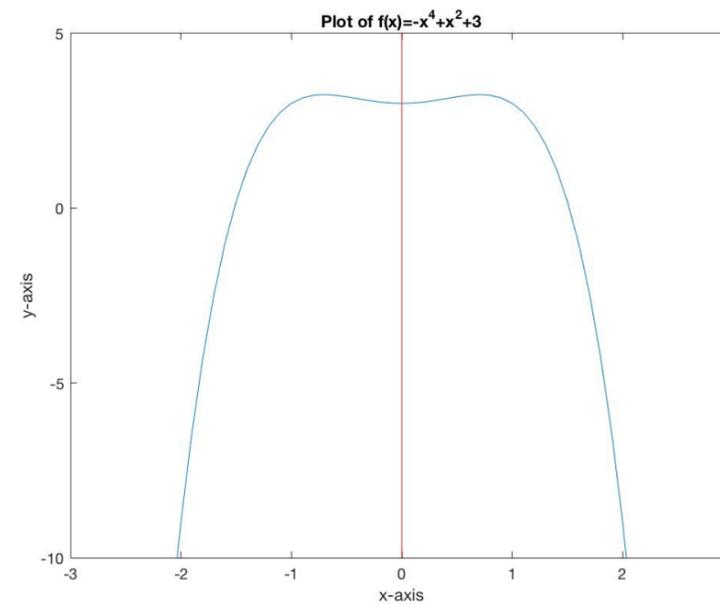
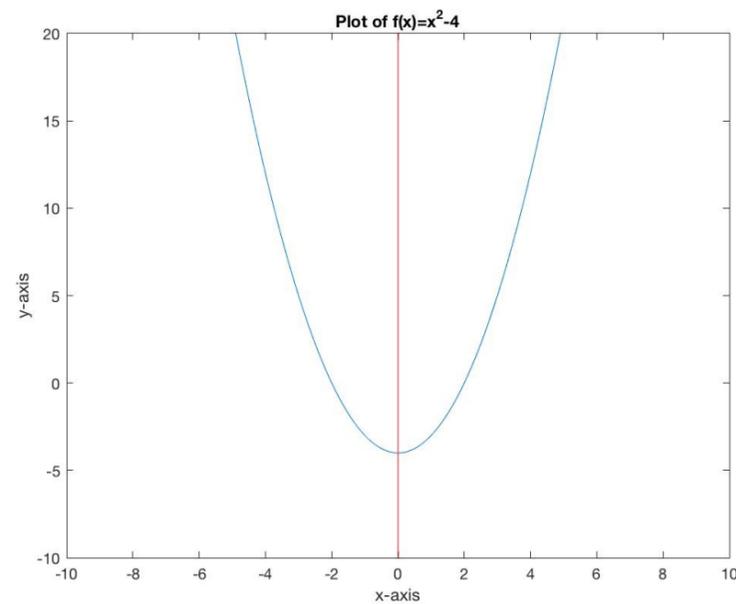
$$(f \circ g)(x) \neq (g \circ f)(x)$$

3.5.1 Plotting Functions

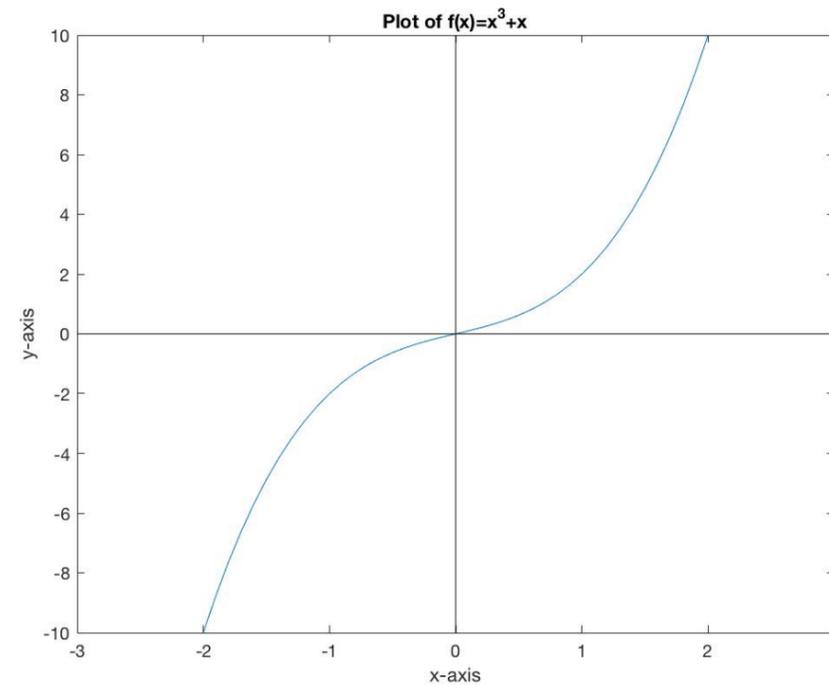
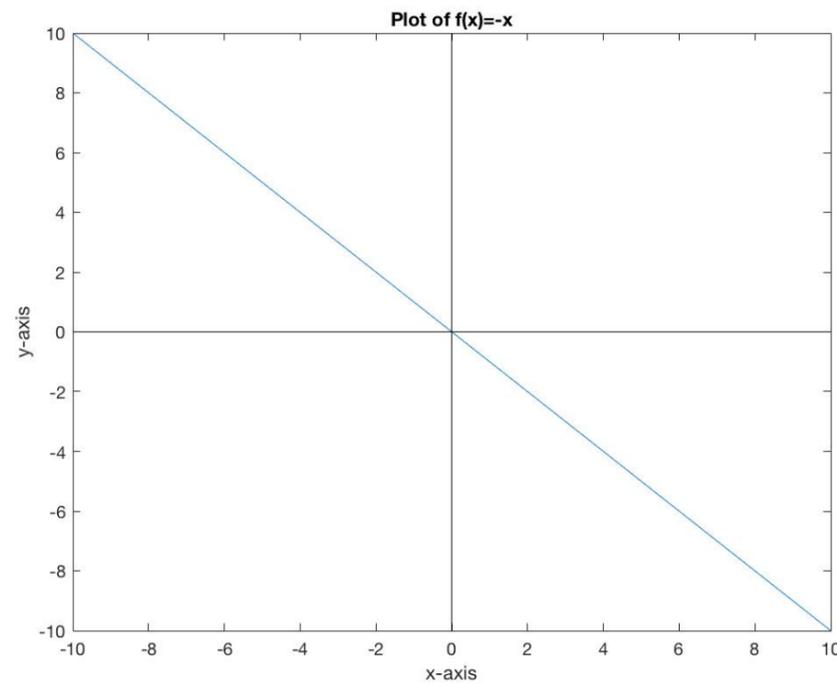
- **Drawing a function in the Cartesian plane is extremely useful in understand the relationship it defines.**
- **One can always attempt to plot a function by computing many pairs $(x, f(x))$, and plotting these on the Cartesian plane.**
- **However, simpler qualitative observations may be more efficient. We will discuss of a few of these notions before moving on to some standard function plots to know.**

3.5.2 Symmetry of Functions

- A function $f(x)$ is said to be *even/is symmetric about the y-axis* if for all values of x , $f(x) = f(-x)$.
- Functions that are even are mirror images of themselves across the y -axis.



- A function $f(x)$ is said to be **odd/has symmetry about the origin** if for all values of x , $f(-x) = -f(x)$
- Functions that are odd can be reflected over the x - axis, then the y -axis.



3.5.3 Transformation of Functions

It is also convenient to consider some standard transformations for functions, and how they manifest visually:

- $f(x) \mapsto f(x + a)$ **shifts the function to the left by a if a is positive, and to the right by a if a is negative.**
- $f(x) \mapsto f(x) + b$ **shifts the function up by b if b is positive, and down by b if b is negative.**
- $f(x) \mapsto f(-x)$ **reflects the function over the y -axis.**
- $f(x) \mapsto -f(x)$ **reflects the function over the x -axis.**

3.6.0 Inverse Functions

Let $f(x)$ be a function. The *inverse function* f is the function that “undoes” $f(x)$; it is denoted $f^{-1}(x)$.

More precisely, for all x in the domain of $f(x)$,

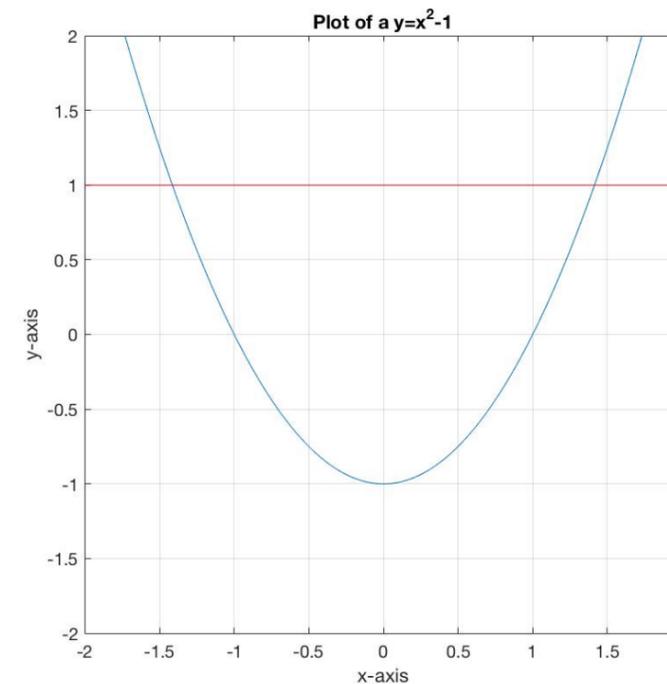
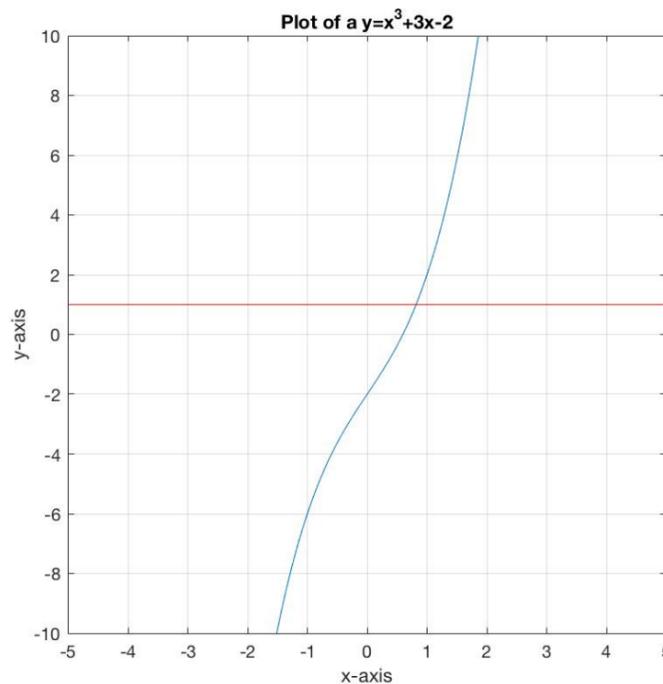
$$(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$$

3.6.1 Remarks on Inverse Functions

- **Not all functions have inverse functions; we will show how to check this shortly.**
- **Note that $f^{-1}(x) \neq (f(x))^{-1}$, that is, inverse functions are not the same as the reciprocal of a function. The notation is subtle.**
- **The domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$.**

3.6.2 Horizontal Line Test

- Recall that one can check if a plot in the Cartesian plane is the plot of a function via the *vertical line test*.
- One can check whether a function $f(x)$ has an inverse function via the *horizontal line test*: the function has an inverse if every horizontal line intersects the plot of $f(x)$ at most once.



4.1.1 Real Numbers

- For us, real numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers.
- There are many subsets of real numbers that are familiar to us.
- Most quantities used to describe things in the world may be understood as real numbers.

$1, 2, 3, \dots$
 $1/5, -1/5, 3/7, \dots$
 $\pi, \sqrt{2}, \log_2(5), \dots$

4.1.2 Integers

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

..., -2, -1, 0, 1, 2, ...

ex: -7 ✓

7/2 ✗

-14/2 ✓

~~Zahl~~
= Zahl → number

4.1.3 Rational Numbers

- Rational numbers are *ratios/fractions* of integers.
- Any number of the form $\frac{p}{q}$ for p, q , integers is rational.
- In particular, every integer is also considered a rational number.
- One must take care: $q = 0$ is not permitted, as this involves division by 0.
- ★ Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.

★ $\frac{p}{0}$ is not defined! many representations: $\frac{1}{2} = \frac{2}{4}$

ex: $\frac{1}{2}$ ✓
 $-\frac{8}{17}$ ✓
 $\frac{2}{1} = 2$ ✓

Remark:

Rationals have many representations: $\frac{1}{2} = \frac{2}{4}$

4.1.4 Irrational Numbers

- There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q .
- Such numbers are called *irrational*; there are many of them.
- Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$
- These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat*.

Ex: Classify the following:

• $\sqrt{11}$

• $\sqrt[3]{11}$

• $\sqrt{11} + 2$

• $\sqrt{2}$

• 17

• -3

• $\sqrt[3]{8}$

• $1 + \frac{1}{5}$

• $\frac{2}{3} + \frac{4}{3}$

• $\sqrt{11}^0$

$$\cdot \pi^2 + \frac{1}{\pi^2}$$

$$\cdot \sqrt{49}$$

$$\cdot \left(\frac{1}{2}\right)^{100}$$

$$\cdot (-1)^{106}$$

4.2.1 Complex Numbers

**James Murphy, Ph.D.
Johns Hopkins University**

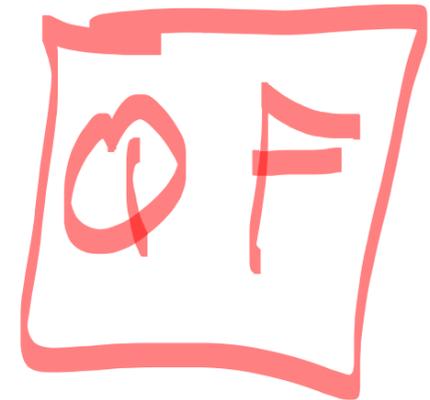
4.2.1 Complex Numbers

- Complex numbers extend the real numbers by introducing the imaginary unit $i = \sqrt{-1}$.
- A complex number is of the form $a + bi$, where a, b are real numbers.
- Complex numbers appear naturally in the context of roots of quadratic polynomials. Recall that

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then the roots are complex.

discriminant



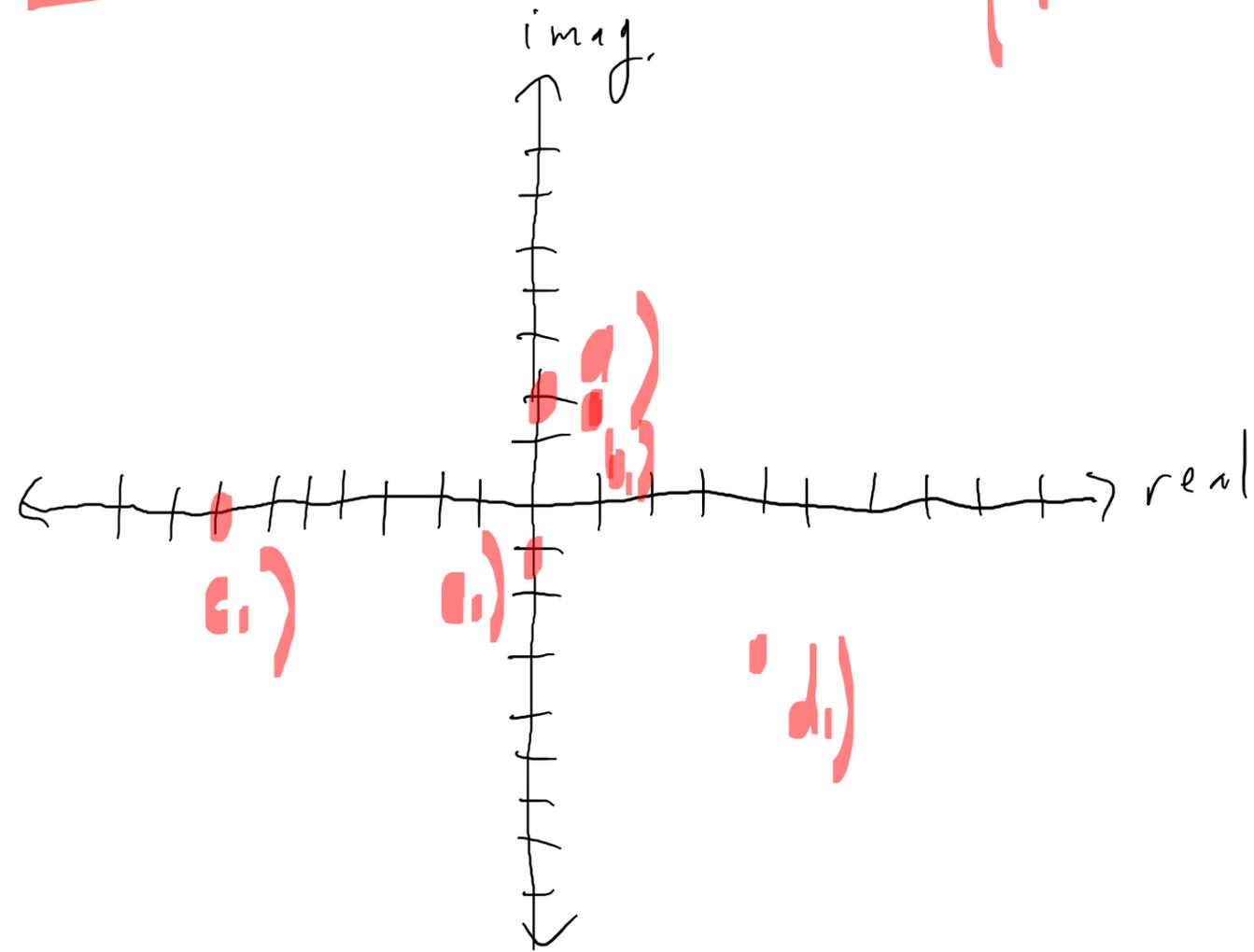
ex:
• $1 + i$
• $2 - 7i$
• $3i$
• 4

$x^2 + 1 = 0$
i.e. $x^2 + 1 = 0$
has $\pm i$
as its
roots

ex: Plot each of the following in

The complex plane

like Cartesian plane



a.) $2i$

b.) $2i + 1$

c.) -7

d.) $4 - 3i$

e.) $-i$

x-axis



real axis

y-axis



imaginary axis

4.2.2 Arithmetic with Complex Numbers

- **Adding and subtracting complex numbers is straightforward:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- **Multiplying requires using the fact that $i^2 = -1$:**

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

c : Simplify each of the following:

$$a.) (3+i)(-2-i)$$

$$b.) (4+2i)^2$$

$$c.) (1+i)^2(1-i)$$

$$d.) i(i+1)$$

4.2.3 Division of Complex Numbers

$$\begin{aligned} \text{ex: } \overline{1-2i} \\ = 1+2i \end{aligned}$$

- Dividing by complex numbers is also tricky. It is convenient to introduce the *conjugate* of a complex number:

$$\overline{a + bi} = a - bi$$

- With this, we may write

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \end{aligned}$$

Remove
imaginary
part

switch sign
of imaginary
part

multiply
top
and bottom
by $c - di = c - di$

ex: Simplify, by removing the imaginary part:

$$\begin{aligned} \text{a.) } \frac{1}{i+1} &= \frac{-i+1}{(i+1)(-i+1)} \\ &= \frac{-i+1}{-i^2+1} \\ &= \frac{-i+1}{2} \end{aligned}$$

$$\begin{aligned} \text{b.) } \frac{2i-1}{3i+2} &= \frac{(2i-1)(-3i+2)}{(3i+2)(-3i+2)} \\ &= \frac{-6i^2+6i+3i-2}{-9i^2+4} \\ &= \frac{9i+4}{13} \end{aligned}$$

$$\begin{aligned} \text{c.) } \frac{2}{2i+4} &= \frac{2(-2i+4)}{(2i+4)(-2i+4)} \\ &= \frac{-4i+8}{-4i^2+16} \\ &= \frac{-4i+8}{20} \\ &= \frac{-i+2}{5} \end{aligned}$$

4.3.1 Sequences and Series

4.3.1 Sequences and Series

- ***Sequences*** are lists of numbers in a given order.
- ***Series*** are sums of sequences.
- Both sequences and series may be finite or infinite.
- Sequences and series have applications to all fields of science and engineering, and are invaluable tools in finance and business.
- We will look at some special examples of these objects, as opposed to stating a general theory.
- Calculus is the home of the general theory of sequences and series.

$a_1 = 1$
 $a_2 = \frac{3}{2}$
 $a_3 = 2$
 $a_4 = \frac{5}{2}$
 \vdots

4.3.2 Notation

- A sequence of numbers is usually written of the form

$$a_1, a_2, a_3, \dots, a_n = \{a_k\}_{k=1}^n$$

$$\begin{aligned} k=1 &\rightarrow a_1 \\ k=2 &\rightarrow a_2 \\ &\vdots \\ k=n &\rightarrow a_n \end{aligned}$$

- This indicates the order, via the subscript, as well as the number of elements in the sequence.
- A series is denoted

$$a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k$$

- The Greek letter Σ is a capital sigma, and it tells us we will sum everything up.

4.3.3 Arithmetic Series

ex: 1, 3, 5, 7, 9, 11, ... $a_k = 2k - 1$

ex: 3, 6, 9, 12, 15, ... $a_k = 3k$

- Many special types of series exist.
- One simple one is *arithmetic series* of the form

$$\sum_{k=1}^n ak + b$$

a, b fixed/constant
 k changes

- Here, a, b are fixed. This series has the nice property that the differences between terms are fixed at a .
- A classical mathematical exercise in proof-based mathematics is to show that if $\{a_k\}_{k=1}^n$ is an arithmetic sequence as above, then

$$\sum_{k=1}^n a_k = \frac{n(a_1 + a_n)}{2}$$

average of first and last element

ex: $\sum_{k=1}^{100} k = 100 \cdot \frac{(1+100)}{2}$

Gauss

$$= \frac{100 \cdot 101}{2}$$

$$= 50 \cdot 101$$

$$= 5050 \quad \checkmark$$

$$\text{ex: } \sum_{j=1}^{50} (2j-1) = 1 + 3 + 5 + 7 + \dots$$

$$= 50 \cdot \left(\frac{1 + 99}{2} \right)$$

$$= 50 \cdot \frac{100}{2}$$

$$= 50 \cdot 50 \\ = 2500 \quad \checkmark$$

$$\text{ex: } \sum_{j=1}^N j = N \cdot \frac{1+N}{2}$$

$$N \text{ is variable} = \frac{N(N+1)}{2}$$

4.3.4 Geometric Series

ex: $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$

- Another important special series is the *geometric series* $\sum_{k=1}^n r^k$ for some $r < 1$.

- There are nice formulas for both the finite and infinite versions of the geometric series:

$$\sum_{k=1}^n r^k = \frac{r(1-r^n)}{1-r} \qquad \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$

ex $= \sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$ $\frac{1}{2} < 1 \Rightarrow$ geo. Series apply

$$= \frac{\frac{1}{2} \cdot \left(1 - \frac{1}{2}^{10}\right)}{1 - \frac{1}{2}}$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \left(1 - \frac{1}{1024}\right)}{\cancel{\frac{1}{2}}} = \frac{1023}{1024} \approx 1$$

ex: $\sum_{j=1}^{\infty} \left(\frac{1}{3}\right)^j \approx \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \frac{1}{19683} + \frac{1}{59049} + \frac{1}{177147} + \frac{1}{531441} + \frac{1}{1594323} + \frac{1}{4782969} + \frac{1}{14348907} + \frac{1}{43046721} + \frac{1}{129140163} + \frac{1}{387420489} + \frac{1}{1162261467} + \frac{1}{3486784401} + \frac{1}{10460353203} + \frac{1}{31381059609} + \frac{1}{94143178827} + \frac{1}{282429536481} + \frac{1}{847288609443} + \frac{1}{2541865828329} + \frac{1}{7625597484987} + \frac{1}{22876792454961} + \frac{1}{68630377364883} + \frac{1}{205891132094649} + \frac{1}{617673396283947} + \frac{1}{1853020188851841} + \frac{1}{5559060566555523} + \frac{1}{16677181699666569} + \frac{1}{50031545098999707} + \frac{1}{150094635296999121} + \frac{1}{450283905890997363} + \frac{1}{1350851717672992089} + \frac{1}{4052555153018976267} + \frac{1}{12157665459056928801} + 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\frac{1}{8019707845470147952363280583241344271318675875862360573430114641273550219177846321307} + \frac{1}{24059123536410443857089841749724032813956027627587081720290343923810550657532538963921} + \frac{1}{72177370609231331571269525249172098441868082882761245160871031771431651972597616881763} + \frac{1}{216532111827693994713808575747516295325604248648283735482613095314294955917792850645289} + \frac{1}{649596335483081984141425727242548885976812745944851206447839285942884867753278551935867} + \frac{1}{1948789006449245952424277181727646657930438237834553619343517857828654603259835655807601} + \frac{1}{5846367019347737857272831545182939973791314713503660858030553573485963809779506967412803} + \frac{1}{17539091058043213571818494635548819921373944140510982574091660720457891429338520902238409} + \frac{1}{52617273174129640715455483906646459764121832421532947722274982161373474287915562706715207} + \frac{1}{157851819522388922146366451719939379292365497264598843166824946484120422852746688110145621} + \frac{1}{473555458567166766439099355159818137877096491793796529490474839452361268558239064330436863} + \frac{1}{1420666375701500309317298065479454413631289475381389588471424518357083805674717192998310589} + \frac{1}{4261999127104500927951894196438363240893868426144168765414273555071251417024151578994951767} + \frac{1}{12785997381313502783855682589315039722681605278432506296242820650713754251072454736984855301} + \frac{1}{38357992143940508351567047767945119168044815835297518888728461952141262753157364210952955903} + \frac{1}{115073976431821525054701143303835357504134447505892556666185385856423788259472092632858867709} + \frac{1}{345221929295464575164103429911506072512403342517677669998556157569271364778416277898576603127} + \frac{1}{1035665787886393725492310289734518217537210027553032999995668475007814094325248832695729809381} + \frac{1}{3107000763659181176476930869203554652611630082659098999986905425023442282975746498087289428143} + \frac{1}{932100229097754352943079260761066395783489024$

ex: $\sum_{j=1}^{\infty} 2 \cdot \left(\frac{7}{8}\right)^j$

$$\frac{7}{8} < 1$$

$$= 2 \sum_{\text{ابتداءً}} \left(\frac{7}{8}\right)^j$$

$$= 2 \cdot \frac{\frac{7}{8}}{1 - \frac{7}{8}}$$

$$= 2 \cdot \frac{\frac{7}{8}}{\frac{1}{8}}$$

$$= 2 \cdot 7 = 14 \quad \checkmark$$

4.4.1 Factorials and Binomial Theorem

- **Counting problems** are among the most important, and challenging, problems in mathematics.
- When discussing the number of combinations or groups possible from some larger set, the notions of **factorial** and **binomial coefficient** play a crucial role.

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

$$\binom{n}{k} = \frac{n!}{(n - k)!k!}$$

4.4.2 Factorial!

- **The factorial of a number is simply the product of itself with all positive integers less than it.**
- **By convention, $0! = 1$.**
- **It is possible to define the factorial for non-integers, but this quite advanced and is not part of the CLEP.**
- **When computing with factorials, it is helpful to *write out the multiplication explicitly*, as there are often cancellations to be made.**

ex: compute

a.) $\binom{5}{3}$

b.) $2!$

c.) $\binom{n}{n-1}$

$$d.) \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$e.) \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

4.4.3 Counting with Factorials

- Factorials are useful for *combinatorics*, i.e. problems involving counting.

- Given n objects, the number of groups of size k when order doesn't matter is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

binomial
coefficient

- Given n objects, the number of groups of size k when order matters is

$$\frac{n!}{(n-k)!} \geq \binom{n}{k}$$

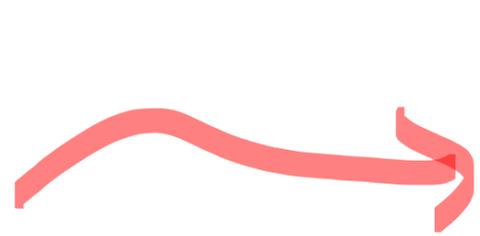
ex: 10 people are at a dance. How many ways to choose pairs?

pairs:

$$n = 10$$

$$k = 2$$

order doesn't matter



$$\binom{10}{2}$$

$$= \frac{10!}{(10-2)! \cdot 2!}$$
$$= \frac{10!}{8! \cdot 2!}$$

$$= \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!} \cdot 2}$$
$$= \frac{90}{2} = 45$$

ex: a.) How many unique order triples of letters exist if each letter may be used only once?

$n = 26$ (corresponds to # letters)

$k = 3$

order matters:

$$\frac{26!}{(26-3)!} = \frac{26!}{23!} = 26 \cdot 25 \cdot 24 = 15600$$

b.) What if order doesn't matter?

$n = 26$

$k = 3$

order doesn't matter:

$$\binom{26}{3} = \frac{26!}{23! \cdot 3!} = 2600$$

4.4.4 Binomial Theorem

ex: $(x+1)^2 = x^2 + 2x + 1$

ex: $(x+1)^{10}$ is hard to do without binomial theorem

- Another important counting application of factorials is in determining the expansion of the binomial:

$(x+y)^n$

$(x+1)^2$

$n=2$
 $y=1$

- The *binomial theorem* states

$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$

binomial coefficient

expand?

$= \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^{n-k} y^k$

ex: Expand $(x-y)^5$ $n=5$

Binomial Theorem $\rightarrow \sum_{k=0}^n x^k (-y)^{n-k} \cdot \binom{n}{k}$

$$\begin{aligned} &= x^0 (-y)^5 \binom{5}{0} + x^1 (-y)^4 \binom{5}{1} + x^2 (-y)^3 \binom{5}{2} + x^3 (-y)^2 \binom{5}{3} \\ &+ x^4 (-y)^1 \binom{5}{4} + x^5 (-y)^0 \binom{5}{5} \\ &= -y^5 + xy^4 \cdot 5 - x^2 y^3 \cdot 10 + x^3 y^2 \cdot 10 + x^4 y \cdot 5 + x^5 \end{aligned}$$

ex: Find the 13th term of $(a+b)^{27}$

$$(a+b)^{27} = \sum_{k=0}^{27} \binom{27}{k} a^k b^{27-k}$$

$k=12$
→

$$\binom{27}{12} a^{12} b^{27-12}$$

$$= \binom{27}{12} a^{12} b^{15}$$

some large number that
we won't compute

4.5.1 Matrices

ex: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$

ex: $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 3 \\ 4 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
 is 2×3

ex: $(1, 0, 5)$

ex: (1)

- A matrix is an array of numbers.
- A matrix M is $m \times n$ if it has m rows and n columns.
- Matrices with only one row or one column are sometimes called **vectors**.
- A matrix with one row and one column is just a number!
- Note that an $m \times n$ matrix has $m \cdot n$ entries.



ex : State

The dimension:

a.) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ 2×2

b.) $(-1, -1, 0)$ 1×3

c.) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2×1

d.) $\begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ -7 & 4 & 6 & 1 \end{pmatrix}$ 3×4

e.) (0) 1×1

f.) ~~1×2~~ $(1, 2)$ 1×2

4.5.2 Matrix Algebra

$$2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 0 \\ 2 \cdot 0 & 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

same # rows
same # columns

- Matrices may be added and subtracted only if they are the same size. In this case, one simply adds or subtracts the corresponding entries of the matrix.
- Matrices may also be multiplied by single numbers; this is called scalar multiplication, and has the impact of multiplying every entry by the scalar.
- Multiplying and dividing matrices is also possible, and is extremely important in modern mathematics and engineering. It is subtle, and not likely to appear on CLEP.



ex : Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - 2 \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ 0 & 1 \end{pmatrix}$$

ex: Compute

$$2 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 12 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 15 \\ -3 \end{pmatrix}$$

4.5.3 Determinant

- 1
- The determinant is a number corresponding to a matrix.
 - It contains important information related to matrix division and using matrices to solve linear systems of equations.
 - It is defined for *square matrices*, i.e. matrices with the same number of rows as columns.
 - For a 2×2 matrix, the formula is:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

matrices

determinant

★ higher number of rows/columns is equivalent

ex: Compute the following determinants:

a.) $\det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 1 \cdot 2 - (0)(-1) = 2 - 0 = 2$

b.) $\det \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = 1 \cdot 2 - (3)(2) = 2 - 6 = -4$

~~c.) $\det \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot 0 - 0 \cdot 2 = 0 - 0 = 0$~~

~~d.) $\det \begin{pmatrix} -3 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} -3 & -8 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 6 \end{pmatrix}$
 $= 24 - 24 = 0$~~