4.1.1 Real Numbers

- For us, real numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers.
- There are many subsets of real numbers that are familiar to us.
- Most quantities used to describe things in the world may be understood as real numbers.

$1, 2, 3, \ldots$

$1/5, -1/5, 3/7, \ldots$

$\pi, \sqrt{2}, \log_2(5), \ldots$

4.1.2 Integers

ex: -7 ✓
$7/2$ ✗
$-14/2$ ✓

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

$$\dots, -2, -1, 0, 1, 2, \dots$$

$\mathbb{Z}$

= Zahl → number

4.1.3 Rational Numbers

- Rational numbers are *ratios/fractions* of integers.
- Any number of the form $\frac{p}{q}$ for $p, q,$ integers is rational.
- In particular, every integer is also considered a rational number.
- One must take care: $q = 0$ is not permitted, as this involves division by 0.
- ☆ Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.

ex: $1/2$ ✓

$.81/17$ ✓

$2/1 = 2$ ✓

Remark:

Rationals have many representations: $\frac{1}{2} = \frac{2}{4}$

☆ $\frac{p}{0}$ is not defined!

4.1.4 Irrational Numbers

- **There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q.**
- **Such numbers are called *irrational*; there are many of them.**
- **Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$**
- **These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat.***

Ex: Classify the following:

- 9π
- $3/77$
- $\pi + 2$
- $\sqrt{2}$
- 17
- -3
- $\sqrt[3]{8}$
- $1 + \frac{1}{5}$
- $\frac{2}{3} + \frac{4}{3}$
- π^0

- $\pi^2 + \frac{1}{\pi^2}$
- $\sqrt{49}$
- $\left(\frac{1}{2}\right)^{100}$
- $(-1)^{100}$

4.2.1 Complex Numbers

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4.2.1 Complex Numbers

ex:
- $1+i$
- $2-7i$
- $3i$
- 4

$i^2+1=0$ i.e. $x^2+1=0$ has $\pm i$ as its roots

- Complex numbers extend the real numbers by introducing the imaginary unit $i=\sqrt{-1}$.
- A complex number is of the form $a+bi$, where a, b are real numbers.
- Complex numbers appear naturally in the context of roots of quadratic polynomials. Recall that

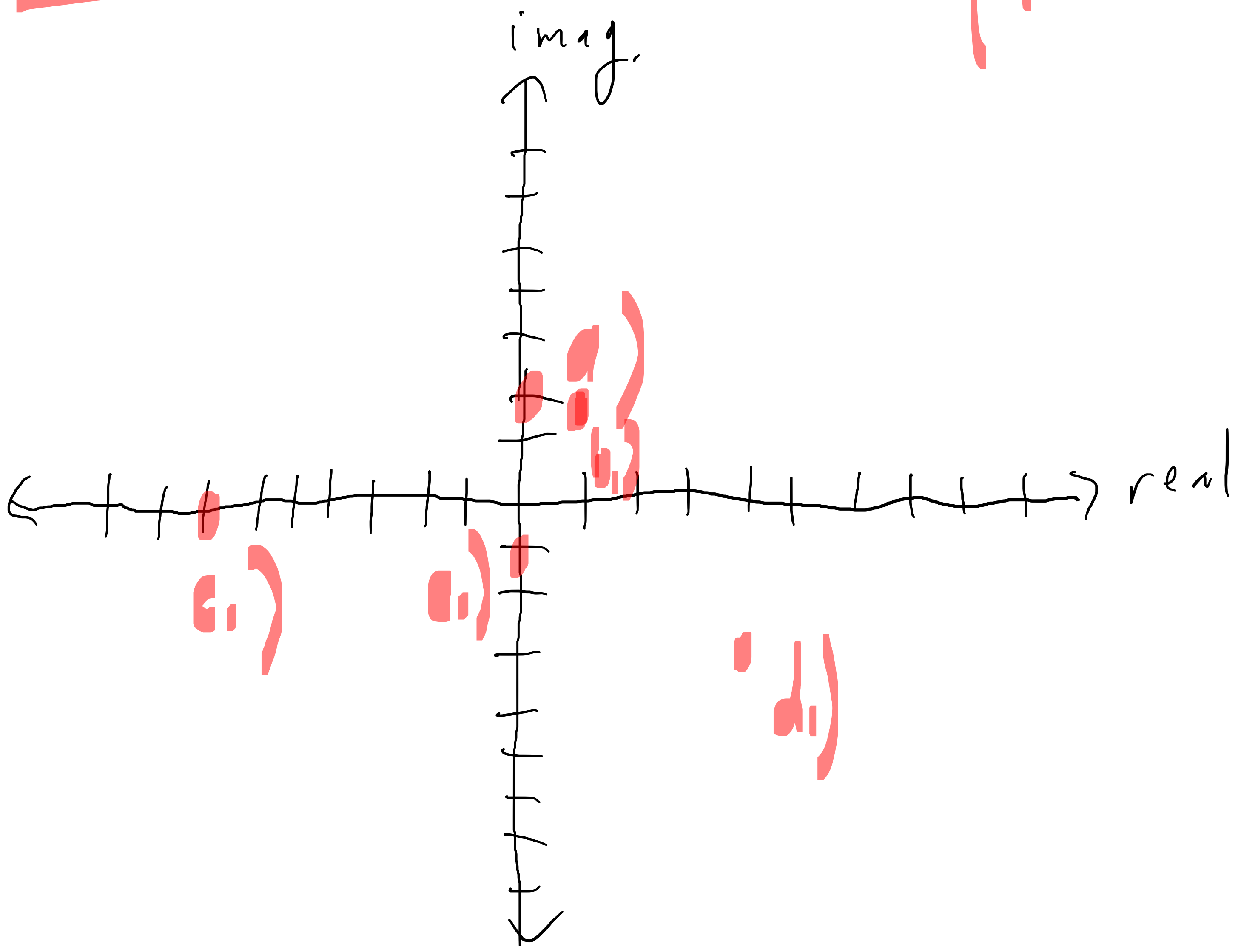
$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\boxed{\text{QF}}$

- If $\underbrace{b^2-4ac}_{\text{discriminant}}<0$, then the roots are complex.

ex: Plot each of the following in the complex plane

like Cartesian plane



a.) $2i$

b.) $2i+1$

c.) -7

d.) $4-3i$

e.) $-i$

x-axis → real axis

y-axis → imaginary axis

4.2.2 Arithmetic with Complex Numbers

- **Adding and subtracting complex numbers is straightforward:**

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

- **Multiplying requires using the fact that** $i^2=-1$ **:**

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2 \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Simplify each of The following:

$(3+i)(-2-i)$ 　　 b.) $(4+2i)^2$

) $(1+i)^2(1-i)$ 　　 d.) $i(i+1)$

4.2.3 Division of Complex Numbers

ex: $\overline{1-2i} = 1+2i$

- **Dividing by complex numbers is also tricky. It is convenient to introduce the *conjugate* of a complex number:**

$$\overline{a+bi} = a - bi$$

switch sign of imaginary part

- **With this, we may write**

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

remove imaginary part

multiply top and bottom by $\overline{c+di} = c-di$

ex: Simplify, by removing the imaginary part:

a.) $\dfrac{1}{i+1}$ $= \dfrac{-i+1}{(i+1)(-i+1)} = \dfrac{-i+1}{-i^2+1} = \dfrac{-i+1}{2}$

b.) $\dfrac{2i-1}{3i+2}$ $= \dfrac{(2i-1)(-3i+2)}{(3i+2)(-3i+2)} = \dfrac{-6i^2+6i+3i-2}{-9i^2+4} = \dfrac{9i+4}{13}$

c.) $\dfrac{2}{2i+4}$ $= \dfrac{2(-2i+4)}{(2i+4)(-2i+4)} = \dfrac{-4i+8}{-4i^2+16} = \dfrac{-4i+8}{20} = \dfrac{-i+2}{5}$

4.3.1 Sequences and Series

4.3.1 Sequences and Series

- *Sequences* are lists of numbers in a given order.
- *Series* are sums of sequences.
- Both sequences and series may be finite or infinite.
- Sequences and series have applications to all fields of science and engineering, and are invaluable tools in finance and business.
- We will look at some special examples of these objects, as opposed to stating a general theory.
- Calculus is the home of the general theory of sequences and series.

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= 3/2 \\ a_3 &= 2 \\ a_4 &= 5/2 \\ &\vdots \end{aligned}\right\}$$

4.3.2 Notation

- A sequence of numbers is usually written of the form

$$a_1, a_2, a_3, ..., a_n = \{a_k\}_{k=1}^{n}$$

$k=1 \to a_1$
$k=2 \to a_2$
$\vdots$
$k=n \to a_n$

- This indicates the order, via the subscript, as well as the number of elements in the sequence.
- A series is denoted

add

$$a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$$

- The Greek letter Σ is a capital sigma, and it tells us we will sum everything up.

4.3.3 Arithmetic Series

ex: 1, 3, 5, 7, 9, 11, ... $a_k = 2k-1$

ex: 3, 6, 9, 12, 15, ... $a_k = 3k$

- Many special types of series exist.
- One simple one is *arithmetic series* of the form

$$\sum_{k=1}^{n} ak + b$$

a, b fixed/constant
k changes

- Here, a, b are fixed. This series has the nice property that the differences between terms are fixed at a .
- A classical mathematical exercise in proof-based mathematics is to show that if $\{a_k\}_{k=1}^{n}$ is an arithmetic sequence as above, then

average first and last element

$$\sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}$$

ex : $\sum_{k=1}^{100} k = 100 \cdot \frac{(1+100)}{2}$

Gauss

$$= \frac{100 \cdot 101}{2}$$

$$= 50 \cdot 101$$

$$= 5050 \checkmark$$

ex: $\sum_{j=1}^{50} (2j-1) = 1+3+5+7+\dots$

$$= 50 \cdot \left(\frac{1+99}{2} \right)$$

$$= 50 \cdot \frac{100}{2}$$

$$= 50 \cdot 50$$

$$= 2500$$

ex: $\sum_{j=1}^{N} j = N \cdot \frac{1+N}{2}$

N is variable $= \frac{N(N+1)}{2}$

4.3.4 Geometric Series

ex: $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$= 1$

- Another important special series is the *geometric series* $\sum_{k=1}^{n} r^k$ for some $r < 1$.

- There are nice formulas for both the finite and infinite versions of the geometric series:

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r} \qquad \sum_{k=1}^{\infty} \frac{r}{1-r}$$

$\underline{ex}$: $\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$ $\frac{1}{2} < 1 \Rightarrow$ geo. series apply

$$= \frac{\frac{1}{2}\left(1-\frac{1}{2}^{10}\right)}{1-\frac{1}{2}}$$

$$= \frac{\cancel{\frac{1}{2}}\cdot\left(1-\frac{1}{1024}\right)}{\cancel{\frac{1}{2}}} = \frac{1023}{1024} \approx 1$$

$\underline{ex}$: $\sum_{j=1}^{\infty} \left(\frac{1}{3}\right)^j \approx \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots \frac{1}{2}$

$$\frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{1}{2} \checkmark$$

$\underline{ex}$: $\sum_{j=1}^{\infty} 2 \cdot \left(\frac{7}{8}\right)^j$ $\quad 7/8 < 1$

$$= 2 \sum_{j=1}^{\infty} \left(\frac{7}{8}\right)^j$$

$$= 2 \cdot \frac{7/8}{1 - 7/8} = 2 \cdot \frac{7/8}{1/8}$$

$$= 2 \cdot 7 = 14 \checkmark$$

4.4.1 Factorials and Binomial Theorem

- **Counting problems** are among the most important, and challenging, problems in mathematics.
- When discussing the number of combinations or groups possible from some larger set, the notions of **factorial** and **binomial coefficient** play a crucial role.

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

4.4.2 Factorial!

- **The factorial of a number is simply the product of itself with all positive integers less than it.**
- **By convention,** $0! = 1$ **.**
- **It is possible to define the factorial for non-integers, but this quite advanced and is not part of the CLEP.**
- **When computing with factorials, it is helpful to *write out the multiplication explicitly*, as there are often cancellations to be made.**

ex: Compute

a.) $\binom{5}{3}$

b.) $2!$

c.) $\binom{n}{n-1}$

d.) $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$

e.) $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

4.4.3 Counting with Factorials

- Factorials are useful for *combinatorics*, i.e. problems involving counting.
- Given n objects, the number of groups of size k when *order doesn't matter* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

.

binomial coefficient

- Given n objects, the number of groups of size k when *order matters* is .

$$\frac{n!}{(n-k)!} \geq \binom{n}{k}$$

ex: 10 people are at a dance. How many ways to choose pairs?

$n = 10$

$k = 2$

order doesn't matter $\longrightarrow$ $\binom{10}{2}$

\# pairs:

$$= \frac{10!}{\underbrace{(10-2)!}_{8!} \cdot 2!} = \frac{10 \cdot 9 \cdot \cancel{8 \cdots 2 \cdot 1}}{\cancel{8 \cdot 7 \cdot 6} \cdots 1 \cdot 2} = \frac{90}{2} = 45$$

ex: a.) How many unique order triples of letters exist if each letter may be used only once?

$n = 26$ (corresponds to # letters)

$k = 3$ order matters: $\frac{26!}{(26-3)!} = \frac{26!}{23!} = \cdots = 26 \cdot 25 \cdot 24 = 15600$

b.) What if order doesn't matter?

$n = 26$

$k = 3$ order doesn't matter: $\binom{26}{3} = \frac{26!}{23! \cdot 3!} = 2600$

4.4.4 Binomial Theorem

ex: $(x+1)^2 = x^2+2x+1$

ex: $(x+1)^{10}$ is hard to do without binomial theorem

- **Another important counting application of factorials is in determining the expansion of the binomial:**

$$(x+y)^n$$

$(x+1)^2$ $n=2$ $y=1$

- **The *binomial theorem* states**

binomial coefficient

expand?

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k$$

ex: Expand $(x-y)^5$, $5 = n$

binomial theorem $\Rightarrow = \sum_{k=0}^{5} x^k (-y)^{n-k} \cdot \binom{n}{k}$

$= x^0(-y)^5\binom{5}{0} + x^1(-y)^4\binom{5}{1} + x^2(-y)^3\binom{5}{2} + x^3(-y)^2\binom{5}{3}$

$+ x^4(-y)^1\binom{5}{4} + x^5(-y)^0\binom{5}{5}$

$= -y^5 + xy^4 \cdot 5 - x^2y^3 \cdot 10 + x^3y^2 \cdot 10 - x^4y \cdot 5 + x^5$

ex: Find the 13^{th} term of $(a+b)^{27}$

$$(a+b)^{27} = \sum_{k=0}^{27} \binom{27}{k} a^k b^{27-k}$$

$k = 12$

$$\binom{27}{12} a^{12} b^{27-12}$$

$$= \underbrace{\binom{27}{12}}_{\text{some large number that we won't compute}} a^{12} b^{15}$$

4.5.1 Matrices

ex:

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

3×3

ex:

$$\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} -1 & 0 & 3 \\ 4 & 6 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

2×3

- A matrix is an array of numbers.
- A matrix M is $m \times n$ if it has m rows and n columns.
- Matrices with only one row or one column are sometimes called *vectors.*
- A matrix with one row and one column is just a number!
- Note that an $m \times n$ matrix has mn entries.

rows columns

ex: $(1, 0, 3)$

ex: (1)

ex : State

The dimension:

a.) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ 2×2

b.) $(-1, -1, 0)$ 1×3

c.) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2×1

d.) $\begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ -7 & 4 & 6 & 1 \end{pmatrix}$ 3×4

e.) (0) 1×1

f.) ~~1x2~~ $(1, 2)$ 1×2

4.5.2 Matrix Algebra

$$2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 & 2\cdot 0 \\ 2\cdot 0 & 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

same # rows
same # columns

- Matrices may be added and subtracted ***only if they are the same size.*** In this case, one simply adds or subtracts the corresponding entries of the matrix.
- Matrices may also be multiplied by single numbers; this is called ***scalar multiplication***, and has the impact of multiplying every entry by the scalar.
- Multiplying and dividing matrices is also possible, and is extremely important in modern mathematics and engineering. It is subtle, and not likely to appear on CLEP.

ex: Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ 0 & 1 \end{pmatrix}$$

ex: Compute

$$2\begin{pmatrix}3\\6\\0\end{pmatrix}+3\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}6\\12\\0\end{pmatrix}+\begin{pmatrix}3\\3\\-3\end{pmatrix}$$

$$=\begin{pmatrix}9\\15\\-3\end{pmatrix}$$

4.5.3 Determinant

matrices

- The determinant is a number corresponding to a matrix.
- It contains important information related to matrix division and using matrices to solve linear systems of equations.
- It is defined for *square matrices*, i.e. matrices with the same number of rows as columns.
- For a 2×2 matrix, the formula is:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

determinants for

★ higher number of rows/columns is [illegible] trickier

<u>ex</u>: Compute the following determinants:

a.) $\det\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 1\cdot 2 - (0)(-1) = 2-0 = 2$

b.) $\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = 1\cdot 2 - (3)(2) = 2-6 = -4$

☆ c.) $\det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1\cdot 0 - 0\cdot 2 = 0-0 = 0$

☆ d.) $\begin{pmatrix} -3 & 6 \\ 4 & -8 \end{pmatrix} = (-3\cdot -8) - (4\cdot 6) = 24-24 = 0$