

4.1.1 Real Numbers

- For us, real numbers are numbers that have no imaginary component. They are in distinction to *imaginary* and *complex* numbers.
- There are many subsets of real numbers that are familiar to us.
- Most quantities used to describe things in the world may be understood as real numbers.

$1, 2, 3, \dots$
 $1/5, -1/5, 3/7, \dots$
 $\pi, \sqrt{2}, \log_2(5), \dots$

4.1.2 Integers

- An important subset of the real numbers are the *integers*.
- Integers are numbers without decimal or fractional parts, and can be positive or negative.
- The number 0 is considered an integer.
- So, the integers may be enumerated as

..., -2, -1, 0, 1, 2, ...

ex: -7 ✓

7/2 ✗

-14/2 ✓

~~Zahl~~
= Zahl → number

4.1.3 Rational Numbers

- Rational numbers are *ratios/fractions* of integers.
- Any number of the form $\frac{p}{q}$ for p, q , integers is rational.
- In particular, every integer is also considered a rational number.
- One must take care: $q = 0$ is not permitted, as this involves division by 0.
- Listing all the rational numbers is trickier than listing all the integers, but it can be done; see Cantor's diagonal argument for a famous method.

ex: $\frac{1}{2}$ ✓
 $-\frac{8}{17}$ ✓
 $\frac{2}{1} = 2$ ✓

Remark:

Rationals have many representations: $\frac{1}{2} = \frac{2}{4}$

★ $\frac{p}{0}$ is not defined! many representations: $\frac{1}{2} = \frac{2}{4}$

4.1.4 Irrational Numbers

- There are real numbers that may not be written as $\frac{p}{q}$, for any integers p, q .
- Such numbers are called *irrational*; there are many of them.
- Famous examples include $\sqrt{2} \approx 1.41$ and $\pi \approx 3.141$
- These approximations are just to give us a sense for these numbers. The actual decimal expansions of irrational numbers *never terminate or repeat*.

Ex: Classify the following:

• $\sqrt{11}$

• $\sqrt[3]{11}$

• $\sqrt{11} + 2$

• $\sqrt{2}$

• 17

• -3

• $\sqrt[3]{8}$

• $1 + \frac{1}{5}$

• $\frac{2}{3} + \frac{4}{3}$

• $\sqrt{11}^0$

$$\cdot \pi^2 + \frac{1}{\pi^2}$$

$$\cdot \sqrt{49}$$

$$\cdot \left(\frac{1}{2}\right)^{100}$$

$$\cdot (-1)^{106}$$

4.2.1 Complex Numbers

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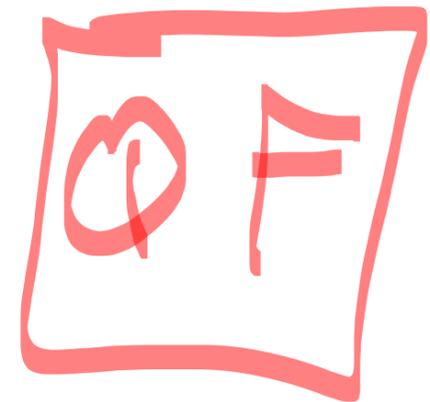
4.2.1 Complex Numbers

- Complex numbers extend the real numbers by introducing the imaginary unit $i = \sqrt{-1}$.
- A complex number is of the form $a + bi$, where a, b are real numbers.
- Complex numbers appear naturally in the context of roots of quadratic polynomials. Recall that

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then the roots are complex.

discriminant



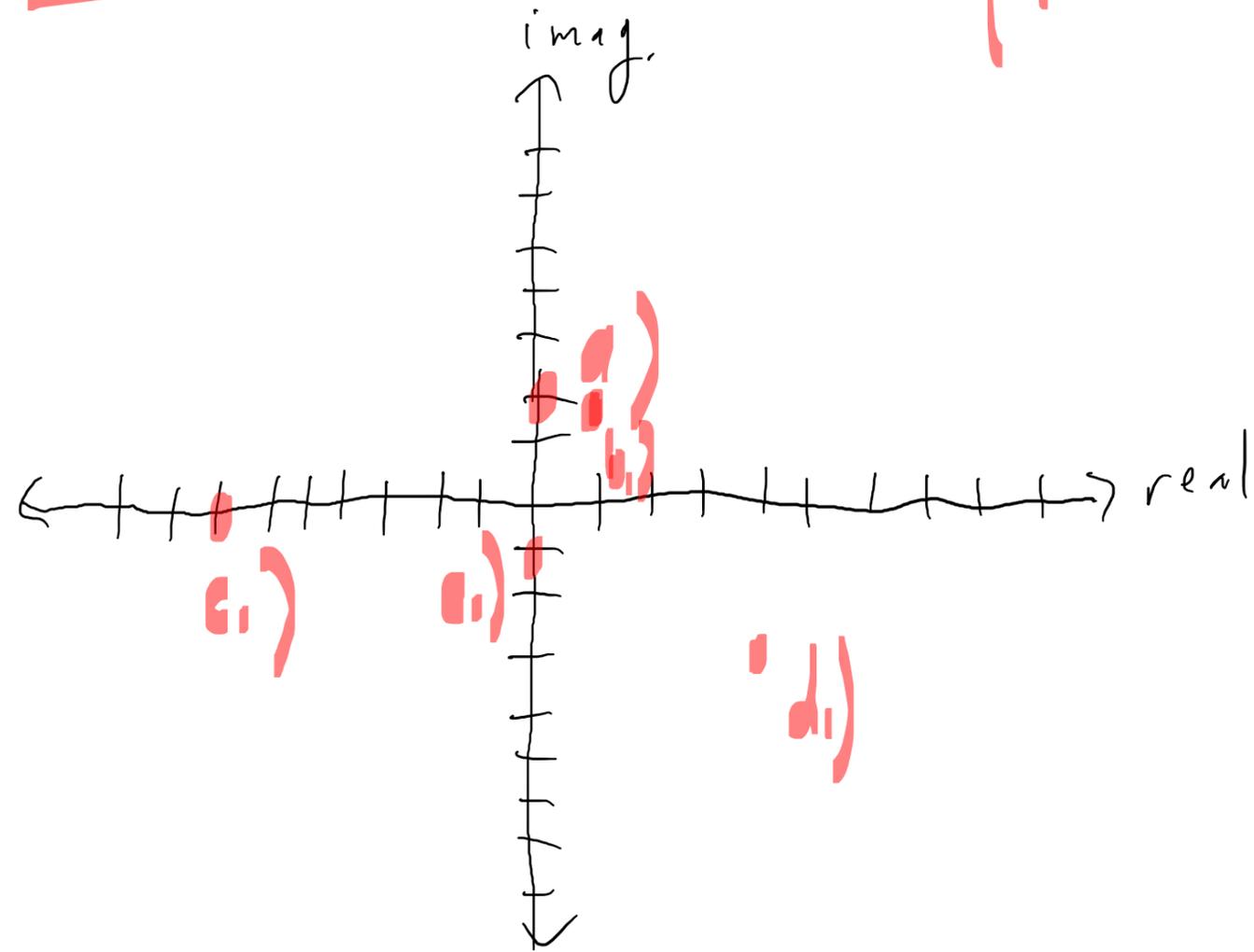
ex:
 $1+i$
 $2-7i$
 $3i$
 4

$x^2 + 1 = 0$
i.e. $x^2 + 1 = 0$
has $\pm i$
as its
roots

ex: Plot each of the following in

The complex plane

like Cartesian
plane



a.) $2i$

b.) $2i + 1$

c.) -7

d.) $4 - 3i$

e.) $-i$

x-axis



real axis

y-axis



imaginary axis

4.2.2 Arithmetic with Complex Numbers

- **Adding and subtracting complex numbers is straightforward:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- **Multiplying requires using the fact that $i^2 = -1$:**

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

c : Simplify each of the following:

$$a.) (3+i)(-2-i)$$

$$b.) (4+2i)^2$$

$$c.) (1+i)^2(1-i)$$

$$d.) i(i+1)$$

4.2.3 Division of Complex Numbers

$$\begin{aligned} \text{ex: } \overline{1-2i} \\ = 1+2i \end{aligned}$$

- Dividing by complex numbers is also tricky. It is convenient to introduce the *conjugate* of a complex number:

$$\overline{a + bi} = a - bi$$

- With this, we may write

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \end{aligned}$$

Remove imaginary part

switch sign of imaginary part

multiply top and bottom by $c - di = \overline{c + di}$

ex: Simplify, by removing the imaginary part:

$$\begin{aligned} \text{a.) } \frac{1}{i+1} &= \frac{-i+1}{(i+1)(-i+1)} \\ &= \frac{-i+1}{-i^2+1} \\ &= \frac{-i+1}{2} \end{aligned}$$

$$\begin{aligned} \text{b.) } \frac{2i-1}{3i+2} &= \frac{(2i-1)(-3i+2)}{(3i+2)(-3i+2)} \\ &= \frac{-6i^2+6i+3i-2}{-9i^2+4} \\ &= \frac{9i+4}{13} \end{aligned}$$

$$\begin{aligned} \text{c.) } \frac{2}{2i+4} &= \frac{2(-2i+4)}{(2i+4)(-2i+4)} \\ &= \frac{-4i+8}{-4i^2+16} \\ &= \frac{-4i+8}{20} \\ &= \frac{-i+2}{5} \end{aligned}$$

4.3.1 Sequences and Series

4.3.1 Sequences and Series

- ***Sequences*** are lists of numbers in a given order.
- ***Series*** are sums of sequences.
- Both sequences and series may be finite or infinite.
- Sequences and series have applications to all fields of science and engineering, and are invaluable tools in finance and business.
- We will look at some special examples of these objects, as opposed to stating a general theory.
- Calculus is the home of the general theory of sequences and series.

Handwritten red annotations showing a sequence of terms:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{3}{2} \\ a_3 &= 2 \\ a_4 &= \frac{5}{2} \\ &\vdots \end{aligned}$$

The terms are grouped by a large red bracket on the right. A red arrow points to the first term, a_1 .

4.3.2 Notation

- A sequence of numbers is usually written of the form

$$a_1, a_2, a_3, \dots, a_n = \{a_k\}_{k=1}^n$$

$$\begin{array}{l} k=1 \rightarrow a_1 \\ k=2 \rightarrow a_2 \\ \vdots \\ k=n \rightarrow a_n \end{array}$$

- This indicates the order, via the subscript, as well as the number of elements in the sequence.
- A series is denoted

$$a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k$$

- The Greek letter Σ is a capital sigma, and it tells us we will sum everything up.

4.3.3 Arithmetic Series

ex: 1, 3, 5, 7, 9, 11, ... $a_k = 2k - 1$

ex: 3, 6, 9, 12, 15, ... $a_k = 3k$

- Many special types of series exist.
- One simple one is *arithmetic series* of the form

$$\sum_{k=1}^n ak + b$$

a, b fixed/constant
 k changes

- Here, a, b are fixed. This series has the nice property that the differences between terms are fixed at a .
- A classical mathematical exercise in proof-based mathematics is to show that if $\{a_k\}_{k=1}^n$ is an arithmetic sequence as above, then

$$\sum_{k=1}^n a_k = \frac{n(a_1 + a_n)}{2}$$

average of first and last element

ex: $\sum_{k=1}^{100} k = 100 \cdot \frac{(1+100)}{2}$

Gauss

$$= \frac{100 \cdot 101}{2}$$

$$= 50 \cdot 101$$

$$= 5050 \quad \checkmark$$

$$\text{ex: } \sum_{j=1}^{50} (2j-1) = 1 + 3 + 5 + 7 + \dots$$

$$= 50 \cdot \left(\frac{1 + 99}{2} \right)$$

$$= 50 \cdot \frac{100}{2}$$

$$= 50 \cdot 50 \\ = 2500 \quad \checkmark$$

$$\text{ex: } \sum_{j=1}^N j = N \cdot \frac{1+N}{2}$$

$$N \text{ is variable} = \frac{N(N+1)}{2}$$

4.3.4 Geometric Series

ex: $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$

- Another important special series is the *geometric series* $\sum_{k=1}^n r^k$ for some $r < 1$.

- There are nice formulas for both the finite and infinite versions of the geometric series:

$$\sum_{k=1}^n r^k = \frac{r(1-r^n)}{1-r} \qquad \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$

ex $= \sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$ $\frac{1}{2} < 1 \Rightarrow$ geo. Series apply

$$= \frac{\frac{1}{2} \cdot \left(1 - \frac{1}{2}^{10}\right)}{1 - \frac{1}{2}}$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \left(1 - \frac{1}{1024}\right)}{\cancel{\frac{1}{2}}} = \frac{1023}{1024} \approx 1$$

ex: $\sum_{j=1}^{\infty} \left(\frac{1}{3}\right)^j \approx \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \frac{1}{19683} + \frac{1}{59049} + \frac{1}{177147} + \frac{1}{531441} + \frac{1}{1594323} + \frac{1}{4782969} + \frac{1}{14348907} + \frac{1}{43046721} + \frac{1}{129140163} + \frac{1}{387420489} + \frac{1}{1162261467} + \frac{1}{3486784401} + \frac{1}{10460353203} + \frac{1}{31381059609} + \frac{1}{94143178827} + \frac{1}{282429536481} + \frac{1}{847288609443} + \frac{1}{2541865828329} + \frac{1}{7625597484987} + \frac{1}{22876792454961} + \frac{1}{68630377364883} + \frac{1}{205891132094649} + \frac{1}{617673396283947} + \frac{1}{1853020188851841} + \frac{1}{5559060566555523} + \frac{1}{16677181699666569} + \frac{1}{50031545098999707} + \frac{1}{150094635296999121} + \frac{1}{450283905890997363} + \frac{1}{1350851717672992089} + \frac{1}{4052555153018976267} + \frac{1}{12157665459056928801} + 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\frac{1}{50256015841064018508565330465220789759539616101809193116096570858548475010590714935399} + \frac{1}{150768047523192055525695991395662369278618848305427579348289712575645425031772144806197} + \frac{1}{452304142569576166577087974186987107835856544916282738044869137726936275095316434418591} + \frac{1}{1356912427708728499731263922560961323507569634748848214134607413180808825285949303255773} + \frac{1}{4070737283126185499193791767682883970522708904246544642403822239542426475857847909767319} + \frac{1}{12212211849378556497581375303048651911568126712739633927211466718627279427573543729301957} + \frac{1}{36636635548135669492744125909145955734704380138218901781634399155881838282720631187905871} + \frac{1}{110009905644407008478232377727437867204113140414656705344903197467645514848161893563717613} + \frac{1}{330029716933221025434697133182313591612339421243969116034709592402936544544485680690752839} + \frac{1}{990089150799663076304091399546940774837018263731907348104128777208809633633456842072258517} + \frac{1}{2970267452399009232912274198640822324511054781195722044312386331626428890899370526216775551} + \frac{1}{8910802357197027698736822595922466973533164343587166132937158994879286672698111578650326653} + \frac{1}{26732407071591083096210467987767400920599493030761508408811476984637859928094334735950979959} + \frac{1}{80197221214773249288631403963302202761798479092284525226434430953913579785282904207879939877} + \frac{1}{240591663644319747865894211889906608285395937276853575679303292861740739355848712623639819631} + \frac{1}{721774990932959243597682635669719824856187811830560727037909878585222218067546137870819458893} + \frac{1}{2165324972798877730793047906009159474568563435491682181113729635755666654202638413612458376679} + \frac{1}{6495974918396633192379143718027478423705690306475046543341188907266999962607915240837375129937} + 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\frac{1}{383578122956312793376796057375804573441400306907045022437753798175209080791731787056260163957635009} + \frac{1}{$

ex: $\sum_{j=1}^{\infty} 2 \cdot \left(\frac{7}{8}\right)^j$

$$\frac{7}{8} < 1$$

$$= 2 \sum_{\text{ابتداءً}} \left(\frac{7}{8}\right)^j$$

$$= 2 \cdot \frac{\frac{7}{8}}{1 - \frac{7}{8}}$$

$$= 2 \cdot \frac{\frac{7}{8}}{\frac{1}{8}}$$

$$= 2 \cdot 7 = 14 \quad \checkmark$$

4.4.1 Factorials and Binomial Theorem

- **Counting problems** are among the most important, and challenging, problems in mathematics.
- When discussing the number of combinations or groups possible from some larger set, the notions of **factorial** and **binomial coefficient** play a crucial role.

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

$$\binom{n}{k} = \frac{n!}{(n - k)!k!}$$

4.4.2 Factorial!

- **The factorial of a number is simply the product of itself with all positive integers less than it.**
- **By convention, $0! = 1$.**
- **It is possible to define the factorial for non-integers, but this quite advanced and is not part of the CLEP.**
- **When computing with factorials, it is helpful to *write out the multiplication explicitly*, as there are often cancellations to be made.**

ex: compute

a.) $\binom{5}{3}$

b.) $2!$

c.) $\binom{n}{n-1}$

$$d.) \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$e.) \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

4.4.3 Counting with Factorials

- Factorials are useful for *combinatorics*, i.e. problems involving counting.

- Given n objects, the number of groups of size k when order doesn't matter is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

binomial
coefficient

- Given n objects, the number of groups of size k when order matters is

$$\frac{n!}{(n-k)!} \geq \binom{n}{k}$$

ex: 10 people are at a dance. How many ways to choose pairs?

pairs:

$$n = 10$$

$$k = 2$$

order doesn't matter

$$\rightarrow \binom{10}{2}$$

$$= \frac{10!}{(10-2)! \cdot 2!}$$
$$= \frac{10!}{8! \cdot 2!}$$

$$= \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!} \cdot 2}$$
$$= \frac{90}{2} = 45$$

ex: a.) How many unique order triples of letters exist if each letter may be used only once?

$n = 26$ (corresponds to # letters)

$k = 3$

order matters:

$$\frac{26!}{(26-3)!} = \frac{26!}{23!} = 26 \cdot 25 \cdot 24 = 15600$$

b.) What if order doesn't matter?

$n = 26$

$k = 3$

order doesn't matter:

$$\binom{26}{3} = \frac{26!}{23! \cdot 3!} = 2600$$

4.4.4 Binomial Theorem

ex: $(x+1)^2 = x^2 + 2x + 1$

ex: $(x+1)^{10}$ is hard to do without binomial theorem

- Another important counting application of factorials is in determining the expansion of the binomial:

$(x+y)^n$

$(x+1)^2$

$n=2$
 $y=1$

- The *binomial theorem* states

$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$

binomial coefficient

expand?

$= \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^{n-k} y^k$

ex: Expand $(x-y)^5$ $n=5$

Binomial Theorem $\rightarrow \sum_{k=0}^n x^k (-y)^{n-k} \cdot \binom{n}{k}$

$$\begin{aligned} &= x^0 (-y)^5 \binom{5}{0} + x^1 (-y)^4 \binom{5}{1} + x^2 (-y)^3 \binom{5}{2} + x^3 (-y)^2 \binom{5}{3} \\ &+ x^4 (-y)^1 \binom{5}{4} + x^5 (-y)^0 \binom{5}{5} \\ &= -y^5 + xy^4 \cdot 5 - x^2 y^3 \cdot 10 + x^3 y^2 \cdot 10 + x^4 y \cdot 5 + x^5 \end{aligned}$$

ex: Find the 13th term of $(a+b)^{27}$

$$(a+b)^{27} = \sum_{k=0}^{27} \binom{27}{k} a^k b^{27-k}$$

$k=12$
→

$$\binom{27}{12} a^{12} b^{27-12}$$

$$= \binom{27}{12} a^{12} b^{15}$$

some large number that
we won't compute

4.5.1 Matrices

ex: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$

ex: $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 3 \\ 4 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
 is $\begin{pmatrix} 2 & 4 & 6 \end{pmatrix}$

ex: $(1, 0, 5)$

ex: (1)

- A matrix is an array of numbers.
- A matrix M is $m \times n$ if it has m rows and n columns.
- Matrices with only one row or one column are sometimes called **vectors**.
- A matrix with one row and one column is just a number!
- Note that an $m \times n$ matrix has $m \cdot n$ entries.



ex : State

The dimension:

a.) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ 2×2

b.) $(-1, -1, 0)$ 1×3

c.) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2×1

d.) $\begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ -7 & 4 & 6 & 1 \end{pmatrix}$ 3×4

e.) (0) 1×1

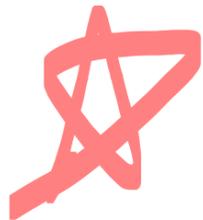
f.) ~~1×2~~ $(1, 2)$ 1×2

4.5.2 Matrix Algebra

$$2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 0 \\ 2 \cdot 0 & 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

same size \neq rows
same \neq columns

- Matrices may be added and subtracted only if they are the same size. In this case, one simply adds or subtracts the corresponding entries of the matrix.
- Matrices may also be multiplied by single numbers; this is called scalar multiplication, and has the impact of multiplying every entry by the scalar.
- Multiplying and dividing matrices is also possible, and is extremely important in modern mathematics and engineering. It is subtle, and not likely to appear on CLEP.



ex : Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} - 2 \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ 0 & 1 \end{pmatrix}$$

ex: Compute

$$2 \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 12 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 15 \\ -3 \end{pmatrix}$$

4.5.3 Determinant

- 1
- The determinant is a number corresponding to a matrix.
 - It contains important information related to matrix division and using matrices to solve linear systems of equations.
 - It is defined for square matrices, i.e. matrices with the same number of rows as columns.
 - For a 2×2 matrix, the formula is:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

matrices

determinant

★ higher number of rows/columns is equivalent

ex: Compute the following determinants:

a.) $\det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 1 \cdot 2 - (0)(-1) = 2 - 0 = 2$

b.) $\det \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = 1 \cdot 2 - (3)(2) = 2 - 6 = -4$

~~c.) $\det \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1 \cdot 0 - 0 \cdot 2 = 0 - 0 = 0$~~

~~d.) $\det \begin{pmatrix} -3 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} -3 & -8 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 6 \end{pmatrix}$
 $= 24 - 24 = 0$~~